# Linear Programs

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A *linear program* ($LP$) consists of *variables* $x = (x_1, \dots, x_n)$, an *objective function* $z(x) = c^T x$ with $c \in \mathbb{Q}^n$ that can either be maximized or minimized, and *constraints* $a^T x \sim b$ with $a \in \mathbb{Q}^n, b \in \mathbb{Q}, \sim \in \{\leq, =, \geq\}$. An LP is in *canonical form* if it is written as

$$\begin{aligned} \text{maximize } & c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ \text{subject to } & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq b_2 \\ & \vdots & \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m \\ & x_1, x_2, \ldots, x_n & \geq 0 \end{aligned}$$

or shorter

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to } & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i \in \{1, \ldots, m\} \\ & x_j \geq 0 & j \in \{1, \ldots, n\} \end{aligned}$$

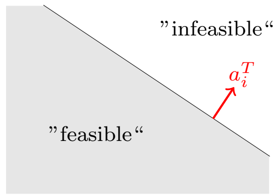
or even shorter

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax \leq b \\ & x \geq 0. \end{aligned}$$

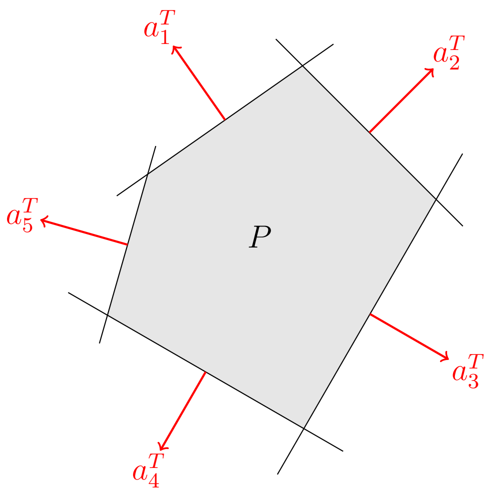
We call the $x_j$ ($j \in \{1, \ldots, n\}$) *decision variables*, $c^T$ the *objective function coefficients*, $A$ the *coefficient matrix*, and $b$ the *right hand side* of the LP. Moreover, we write $a_i^T$ for the coefficients of the $i$-th constraint. Note that every LP has an "equivalent" LP in canonical form:

- $\min_x c^T x = \max_x -c^T x$,
- $a_i^T x \geq b_i$ iff $-a_i^T x \leq -b_i$,
- $a_i^T x = b_i$ iff $a_i^T x \leq b_i$ and $a_i^T x \geq b_i$,
- $x_j \leq 0$ iff $-x_j \geq 0$,
- $x_j$ unbounded: Replace $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$.

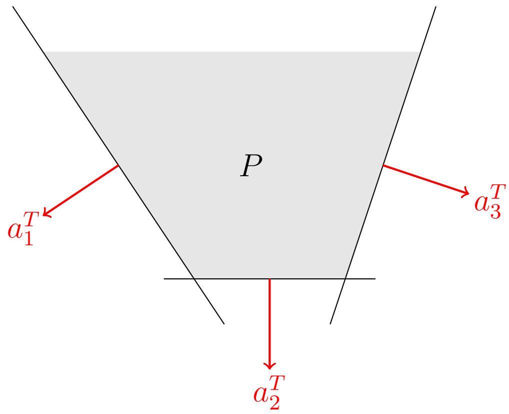
Every constraint (from an LP in canonical form) defines a *halfspace* in $\mathbb{R}^n$ and its separating hyperplane has $a_i^T$ as its normal vector.




We call the set of points that satisfy all constraints *feasible region* $P \coloneqq \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$. $P$ is a *polyhedron*, i.e. the intersection of finitely many halfspaces. If $P$ is bounded, it is called a *polytope*.

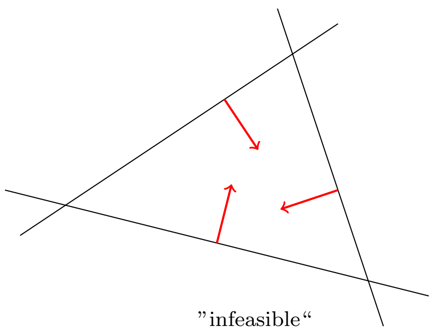



"bounded"

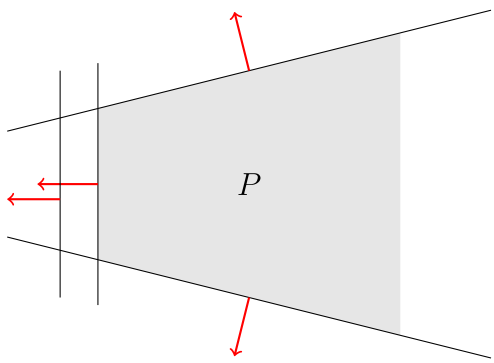



"unbounded"

If the constraints contradict themselves (for instance, $x + y \leq 1$, $x \geq 2$, and $x, y \geq 0$), then $P = \varnothing$, a corresponsing LP is called *infeasible*. Constraints might also be *redundant*, i.e. a constraint can be omitted without increasing the polyhedron (for example, $x+y \geq 0$ and $x, y \geq 0$).



"infeasible"




"redundant"

Polyhedra are *convex*, i.e. they satisfy Jensen's inequality: For all $x, y \in P$ and all $\vartheta \in [0, 1]$, $(1 - \vartheta)x + \vartheta y \in P$.

*Proof.* Let $x, y \in P$ and $\vartheta \in [0, 1]$, then

$$\underbrace{(1-\vartheta)}_{\geq 0}\underbrace{x}_{\geq 0} + \underbrace{\vartheta}_{\geq 0}\underbrace{y}_{\geq 0} \geq 0$$

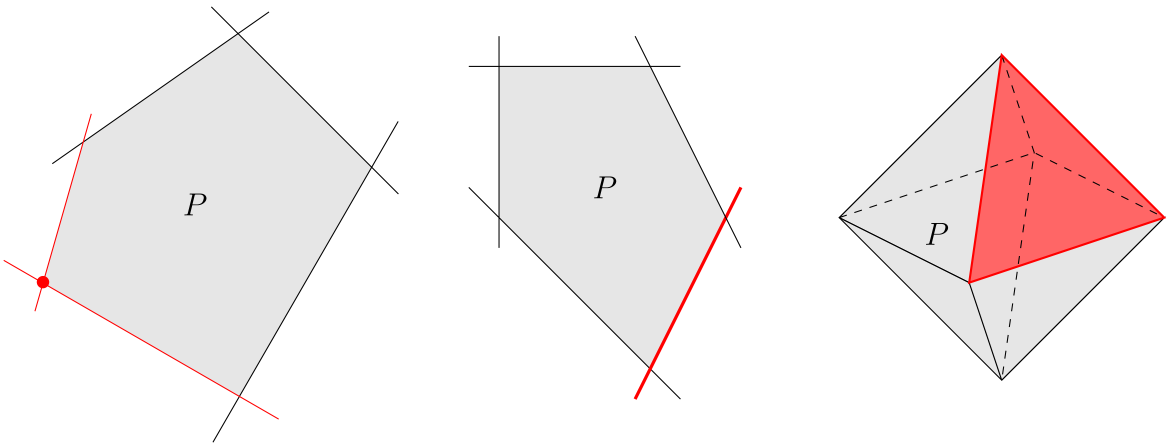
and moreover,

$$
\begin{aligned}
A((1-\vartheta)x+\vartheta y) &= (1-\vartheta)\underbrace{Ax}_{\leq b}+\vartheta\underbrace{Ay}_{\leq b} \\
&\leq (1-\vartheta)b+\vartheta b \\
&= b.
\end{aligned}
$$

□

The *dimension* $\dim P$ is the dimension of the smallest affine subspace containg $P$. If $P \subseteq \mathbb{R}^n$ with $\dim P = n$, then $P$ is called *full dimensional*. For a given $x \in P$, a constraint $a_i^T x \leq b_i$ is called *active* (or *binding*) if $a_i^T x = b_i$. A *face* with respect to $H \subseteq \{1, \dots, m\}$ is

$$F \coloneqq \{x \in P \mid a_i^T x \leq b_i \text{ active in } x, i \in H\}.$$




”vertex (0-dim)“ ”edge (1-dim)“ ”facet ((n − 1)-dim)“

Every face itself is again a polyhedron. A point $x \in P$ is called *feasible solution*, $x^* \in P$ such that $c^T x^* \geq c^T x$ for all $x \in P$ is called *optimal solution* (provided that the objective function should be maximized), $c^T x^*$ is called *optimum*.

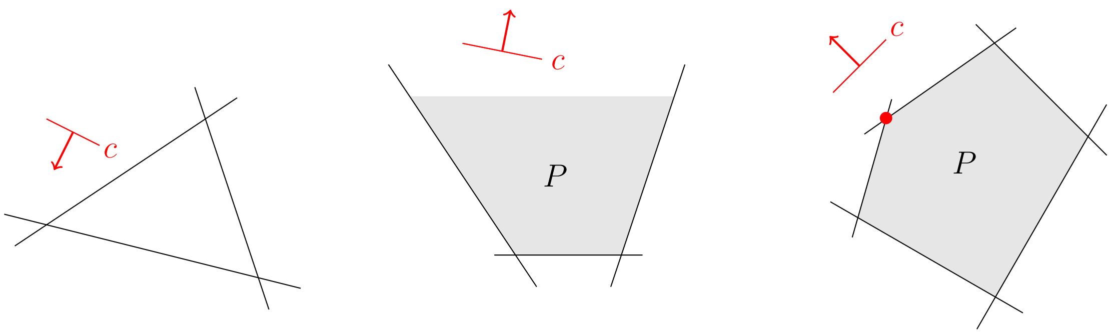
**Theorem 1** (Fundamental Theorem of Linear Programming)**.** *For every LP exactly one of the following hold:*

*(i) The LP is infeasible, i.e.* $P = \varnothing$.

*(ii) The optimum is unbounded, i.e. for all* $M > 0$*, there exists* $x \in P$ *with* $c^T x \geq M$.

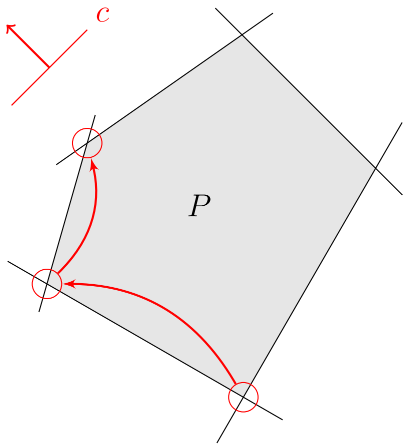
*(iii) There exists a finite optimal solution.*

*If the last case is true, then the optimal sollution is assumed at a vertex of* $P$.




**Lemma 2.** *An* $n$*-dimensional polyhedron given by* $m$ *constraints has at most* $\binom{m}{n}$ *vertices, i.e. finitely many.*

Thus, of all (usually infinitely many) feasible points, only finitely many are relevant. However, enumerating all those points and choosing the best solution (*brute force*) is still very inefficient.
Idea for the Simplex algorithm: Move from vertex to vertex such that the objective value only increases (decreases, respectively, if objective is to minimize).



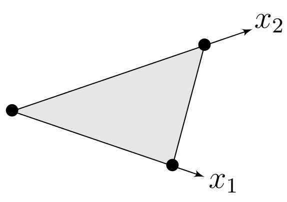

We introduce *slack variables* to fill gaps between constraints and corresponding right hand side.

$$y \coloneqq b - Ax, \quad y \geq 0.$$

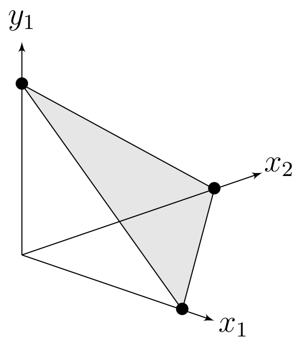
This yields to *standard form* for LPs:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax + y = b \\ &x, y \geq 0 \end{aligned} \qquad \begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax = b \\ &x \geq 0 \end{aligned}$$

where the LP on the right is obtained by setting $x_{n+i} = y_i$ for $i \in \{1, \ldots, m\}$ and we extend $c$ and $A$ in the obvious way.
The transformation from canonical form to standard form preserves dimension and vertices of the polyhedron.




$P = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$




$P' = \{(x, y) \in \mathbb{R}^3_+ \mid x_1 + x_2 + y_1 = 1\}$

Note that the slack variable $y_i = 0$ iff the $i$-th constraint is active. We can also interpret $x$ as slack variable, just note that $x_j = 0$ iff $x_j \geq 0$ active.
Let $A_1, A_2, \ldots, A_n, A_{n+1}, \ldots, A_{n+m}$ be the columns of $A$ (the latter $m$ columns are unit vectors for slack varaiables). For $J \subseteq (1, \ldots, n+m)$, let $A_J$ denote the matrix consisting of columns $A_j$ with $j \in J$, e.g. for

$$A = \begin{pmatrix} 3 & 7 & 0 & -1 & 1 & 0 \\ -1 & -1 & -2 & 2 & 0 & 1 \end{pmatrix}, \quad J = (5, 2) \implies A_J = \begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}.$$

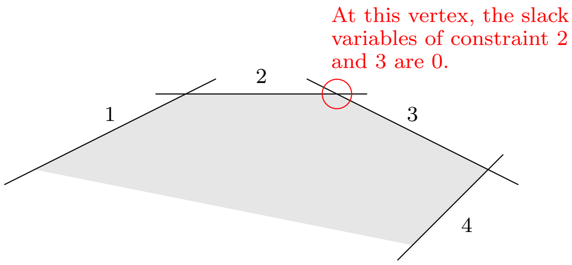
A *basis* $B = (B_1, \dots, B_m) \subseteq (1, \dots, n+m)$ is a subset of $m$ column indices such that the corresponding columns are linearly independent. $N = (1, \dots, n+m) \setminus B$ is called *non basis*. Variables $x_j$ with $j \in B$ are called *basic variables*, and *non-basic variables* if $i \in N$. A vector $x \in \mathbb{Q}^{n+m}$ is a *basic solution* to $Ax = b$, $x \geq 0$ if there is a basis $B$ such that

- $A_B x_B = b$ (*uniqueness*),
- $x_N = 0$ (*at boundary, vertex*).

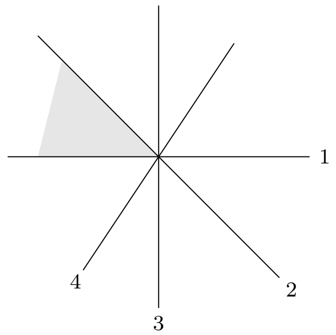
If additionally $x_B \geq 0$ holds $x$ is called *feasible basic solution*.

**Theorem 3.** *Every feasible basic solution corresponds to exactly one vertex of $P$.*

Basic solution are also called *extreme point solutions*.
We need $\dim P$ constraints to describe a vertex. The non-basic variables correspond to the active constraints.




Note that the basis in one vertex is not necessarily unique. We call those vertices *degnerate*.




Possible bases:
$B = (1,2), (1,3), (1,4), (2,3), \dots$

Given a standard form LP with $Ax = b$, $x \geq 0$ and basis $B$:

$$\begin{aligned} A_B x_B + A_N x_N &= b \\ \iff x_B &= \underbrace{A_B^{-1} b}_{=:\bar{b}} - \underbrace{A_B^{-1} A_N}_{=:\bar{A}_N} \underbrace{x_N}_{=0} . \end{aligned}$$

Using the objective function $z(x) = c^T x$:

$$\begin{aligned} z(x) &= c^T x \\ &= c_B^T x_B + c_N^T x_N \\ &= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N \\ &= c_B^T A_B^{-1} - c_B^T A_B^{-1} A_N x_N + c_N^T x_N \\ &= \underbrace{c_B^T A_B^{-1} b}_{=:\bar{z}} + \underbrace{(c_N^T - c_B^T A_B^{-1} A_N)}_{=:\bar{c}_N^T \text{ "reduced cost"}} \underbrace{x_N}_{=0} . \end{aligned}$$

Optimality condition: Basis $B$ (and the corresponding vertex) optimal if reduced cost $\bar{c}_N^T \leq 0$. Intuitively, no non-basic variable can be increased without decreasing the value of $z$.
Otherwise we can find a non-basic variable that can "improve" the objective value. This means we deactivate a constraint (increasing its slack) and move to another vertex:

- Initially: Vertex given by $B, N$.
- If there is a non-basic variable $x_s$, $s \in N$, with $\bar{c}_s > 0$, it is beneficial to increase $x_s$ (currently $x_s = 0$).
- Since $x_B = A_B^{-1}b - A_B^{-1}A_N x_N$, the values of basic-variables decrease if $A_B^{-1}A_s > 0$.
- The maximum value for $x_s$ is determined by "the first" basic variable which becomes 0.
- If this never happens, i.e. $A_B^{-1}A_s \leq 0$, then $x_s$ can be arbitrarily increased; in this case the objective value is unbounded.

The first basis: If $b \geq 0$, all slack variables are a feasible basis, i.e. $B = (n+1, \dots, n+m)$, and hence, $A_B = A_B^{-1} = \mathbb{1}_m$ and $\bar{A}_N = A_N$, $\bar{b} = b$. The first basic solution is then $x_{n+i} = b_i$ for $i \in \{1, \dots, m\}$ and $x_1 = \dots = x_n = 0$ (i.e. the origin). Since $c_B^T = 0$ (all slack), we have $\bar{c}_N^T = C_N^T$ and $\bar{z} = 0$. For calculation by hand, we can store all coefficients in a *dictionary* (or *tableau*):

| | | | |
|---|---|---|---|
| | $\bar{c}_N^T$ | $0$ | $\bar{z}$ |
| $x_B$ | $\bar{A}_N$ | $\mathbb{1}_m$ | $\bar{b}$ |
| | $x_N$ | $x_B$ | |

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**Algorithm 1:** Basic Simplex algorithm

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Compute initial basic feasible solution.
while Current basic solution not optimal do
    Move to best adjacent basic solution.
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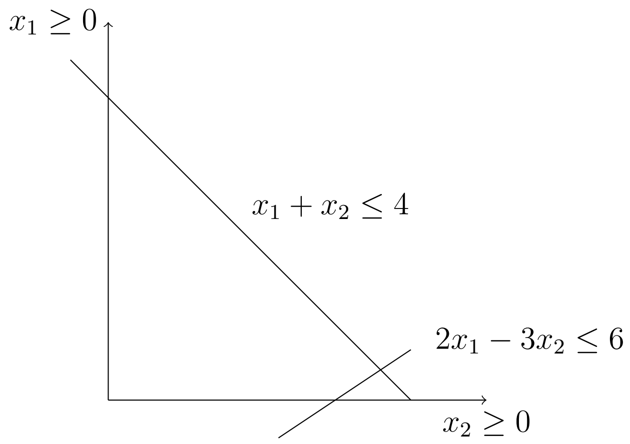
In the following we demonstrate the Simplex method on a small example that captures the whole pivoting routines

**Example.** We consider the LP in canonical form

$$\begin{aligned} \text{maximize } & 2x_1 + x_2 \\ \text{subject to } & 2x_1 - 3x_2 \leq 6 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

The corresponding polytope is sketched in the next figure.

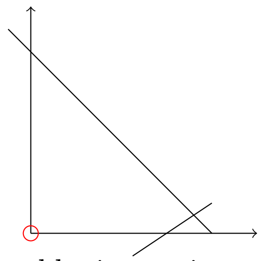


The corresponding LP in standard form is

$$\begin{aligned}
\text{maximize } & 2x_1 + x_2 \\
\text{subject to } & 2x_1 - 3x_2 + x_3 = 6 \\
& x_1 + x_2 + x_4 = 4 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

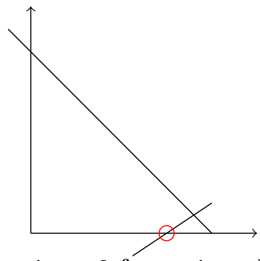
Since $b = (6, 4) \geq (0, 0)$, we can choose the slack variables as basis, i.e. $B = (3, 4)$. Thus, the first basic solution is $(x_1, x_2, x_3, x_4) = (0, 0, 6, 4)$. From the standard form, we construct the initial simplex tableau.

| | | | | | |
|---|---|---|---|---|---|
| | 2 | 1 | 0 | 0 | 0 |
| $x_3$ | **2** | $-3$ | 1 | 0 | 6 |
| $x_4$ | 1 | 1 | 0 | 1 | 4 |
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |



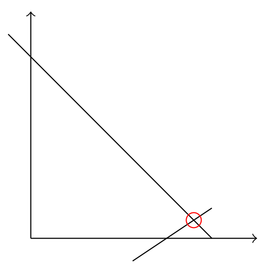
As we can see the greatest improvement of $\bar{z}$ can be obtained by increasing $x_1$ to a positive value ($\bar{c}_1 = 2 > 1 = \bar{c}_2$). Since $x_B = \bar{b} - \bar{A}_1 x_1$, we have to decreas basic variables that have a positive value in $\bar{A}_1 = (2, 1)^T$ (i.e. all of them): Increasing $x_1$ by 1 forces $x_3$ to decrease by 2 and $x_4$ by 1. We can only decrease $x_3, x_4$ until one of them falls to 0, i.e. $x_1$ can only be increased to $\min\{\frac{6}{2}, \frac{4}{1}\} = 3$. Hence, in this case $x_3$ drops to 0. Since $x_1$ becomes a basic variable, its column should become $(1, 0)^T$, so we do similar transformation as in the Gaussian eliminination with the **2** as pivot. Similarly, the reduced cost should become 0.

| | | | | | |
|---|---|---|---|---|---|
| | 0 | 4 | $-1$ | 0 | $-6$ |
| $x_1$ | 1 | $-\frac{3}{2}$ | $\frac{1}{2}$ | 0 | 3 |
| $x_4$ | 0 | $\mathbf{\frac{5}{2}}$ | $-\frac{1}{2}$ | 1 | 1 |
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |



Note that by doing these Gaussian transformations we obtain $-6$ for $\bar{z}$, i.e. the wrong sign but we can live with this technical defect. The current solution now is $(3, 0, 0, 1)$ which can be improved by increasing $x_2$ since $\bar{c}_2 = 4 > 0$. Note that the second constraint is not binding yet, as its slack ist positive. However, the first constraint is binding (we designed it that way by decreasing its slack to 0). Now we use $\mathbf{\frac{5}{2}}$ as pivot as it is the only positive entry in the $x_2$-column.

| | | | | | |
|---|---|---|---|---|---|
| | 0 | 0 | $-\frac{1}{5}$ | $-\frac{8}{5}$ | $-7.4$ |
| $x_1$ | 1 | 0 | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{18}{5}$ |
| $x_2$ | 0 | 1 | $-\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ |
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |



Note since $a_{12} = -\frac{3}{2} < 0$ in the previous tableau, by increasing $x_2$, we actually were able to increase $x_1$ further to $\frac{18}{5} = 3.6$ (observe the first constraint). The reduced cost of any variable are now non-positive; hence, there is no suitable variable to increase and the current solution $(\frac{18}{5}, \frac{2}{5}, 0, 0)$ is optimal.