# Exercise 1

## 1 Index notation

In a right-handed, orthonormal system of fixed basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of unit length, an arbitrary vector $\mathbf{u}$ has the following co-ordinate (component) expression

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = \sum_{i=1}^{3} u_i\mathbf{e}_i = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

By dropping the summation symbol, we get the vector $\mathbf{u}$ in Einstein summation convention as follows

$$\mathbf{u} = u_i\mathbf{e}_i$$

where a summation is implied in an expression, whenever indices occur twice. The repeated indices are called free or dummy indices. In order to evaluate and simplify expressions in vector and tensor algebra we make use of

1. The *Kronecker* delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

2. The *Levi-Cevita* symbol

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i,j,k) \text{ has even permutation of} (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ has odd permutations of} (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

For the dot product of two orthonormal basis vectors we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

> **Note**
>
> In this course, we use the convention that **index notation** signals the use of index notation combined with the Einstein summation convention.

We often require you to change the notation from one form to another.

**Example:**

- **index notation**:

$$a_i + b_i$$

- **vector notation**:

$$\mathbf{a} + \mathbf{b}$$

- **component notation**

$$\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

or

$$\begin{pmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{pmatrix}$$

## Tasks

1. Show that the scalar projection of a vector $\mathbf{u}$ on any of the basis vectors, $\mathrm{proj}_{\mathbf{e}_i}(\mathbf{u}) := \mathbf{u}\cdot\mathbf{e}_i$, is identical to the component of $\mathbf{u}$ corresponding to that basis vector.

2. Write out the dot-product of two arbitrary vectors $\mathbf{u}\cdot\mathbf{v}$ in component notation and index notation.

3. Write the cross-product of two arbitrary vectors $\mathbf{u}\times\mathbf{v}$ in component notation and in index notation.

4. Write the expression $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ in component notation and in index notation.

5. Show that, for three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ the triple vector product

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$$

Tip

Between the *Kronecker* delta and the *Levi-Civita* symbol, the following identity holds

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$