Exercise 2

1 Lagrangian vs Eulerian description

Note

The learning goal of this exercise is to understand how we can switch between the Lagrangian and Eulerian descriptions of **ideal** fluids (e.g. no diffusion). To do so, we need full information about the velocity field $\mathbf{v}(t, \mathbf{x})$ in the Eulerian case or the trajectories $\mathbf{X}(t, \mathbf{x}_0$ for all starting positions $\mathbf{x}_0$ in the Lagrangian case.

Given the trajectories $\mathbf{X}(t; \mathbf{x}_0) = (X(t; \mathbf{x}_0), Y(t; \mathbf{x}_0), Z(t; \mathbf{x}_0))^T$

$$
\begin{aligned}
X(t; \mathbf{x}_0) &= x_0\, cos(t) + y_0\, sin(t) \\
Y(t; \mathbf{x}_0) &= -x_0\, sin(t) + y_0\, cos(t) \\
Z(t; \mathbf{x}_0) &= 0
\end{aligned}
$$

with $\mathbf{x}_0 = (x_0, y_0, z_0)^T$ indicating coordinates at $t = 0$, we now measure a scalar ϕ on the trajectory of all particles $\mathbf{X}(t; \mathbf{x}_0)$ with their respective starting positions $\mathbf{x}_0$:

$$
\phi(t, x = \mathbf{X}(t; \mathbf{x}_0)) = x_0^2 + y_0^2
$$

a)

Sketch two trajectories for $t \in (0, 2\pi)$ as well as their respective field values $\phi(t, x = \mathbf{X}(t; \mathbf{x}_0))$ over time.

partial solution

Use the online tool linked in the lecture notes to verify your trajectory.

b)

Do you know which physical field in fluid flow applications exhibits the same bahavior as ϕ?

> partial solution
>
> The density of incompressible fluids.

c)

We now want to describe $\phi(t, \mathbf{x})$ in the Eulerian frame. The final expression should therefore only depend on the independent variables $(t, \mathbf{x})$.

> partial solution
>
> 1. Write the velocity field as a matrix-vector product $\mathbf{X} = \underline{\underline{R}}\,\mathbf{x_0}$
> 2. We can 'reverse time' and find the the initial position $\mathbf{x}_0$ for each trajectory $\mathbf{X}(t; \mathbf{x}_0)$ at time t.
> 3. Substitute this relation to obtain the scalar field $\phi(t, \mathbf{x})$ in the Eulerian frame.

d)

Compute $\frac{d\phi(t,\mathbf{X}(t;\mathbf{x}_0))}{dt}$ and $\frac{D\phi(t,\mathbf{x})}{dt}$ using their definitions.

> partial solution
>
> $\frac{d\phi(t,\mathbf{X}(t;\mathbf{x}_0)}{dt} = 0$ is trivially computed in the Lagrangian frame. $\frac{D\phi(t,\mathbf{x})}{dt}$ needs to give the same result, as we are only changing our frame of reference! You are on the right track if your velocity field reads $\mathbf{v} = (y, -x)^T$.

e)

Consider experiment where you measure a scalar field ψ on the trajectory of all particles$\psi(t, \mathbf{x} = \mathbf{X}(t; \mathbf{x}_0)) = X^2 + Y^2$ with their respective starting positions $\mathbf{x}_0$. Rewrite ψ in the Eulerian frame.

partial solution

We only require the fact that we currently restrict $\mathbf{x} \stackrel{!}{=} \mathbf{X}(t;\mathbf{x_0})$ to the particle trajectory. As the trajectory has the same form for all particles, we can directly move to the Eulerian frame.

2 Streamlines/Streaklines/Pathlines

Consider the velocity field

$$\mathbf{v}(t,x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -kx + \alpha t \\ ky \end{pmatrix}$$

with k, α being positive constants.

a)

Compute the pathlines.

partial solution

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} C\, e^{-k\,t} - \frac{\alpha}{k^2} + \frac{\alpha\, t}{k} \\ C_2\, e^{k\,t} \end{pmatrix} \text{ with} \\ C &= (\mathbf{x}_0)_x + \frac{\alpha}{k^2} \\ C_2 &= (\mathbf{x}_0)_y \end{aligned}$$

b)

Compute the streamlines.

partial solution

$$\mathbf{X} = \begin{pmatrix} C\ e^{-k\ s} + \frac{\alpha\ t^*}{k} \\ C_2\ e^{k\ s} \end{pmatrix} \text{ with}$$

$$C = (\mathbf{x}_0)_x - \frac{\alpha\ t^*}{k}$$

$$C_2 = (\mathbf{x}_0)_y$$

c)

Compute the streaklines.

partial solution

$$\mathbf{X} = \begin{pmatrix} C\ e^{-k\ \theta} - \frac{\alpha}{k^2} + \frac{\alpha\ (\tau+\theta)}{k} \\ C_2\ e^{k\ t} \end{pmatrix} \text{ with}$$

$$C = (\mathbf{x}_0)_x + \frac{\alpha}{k^2} - \frac{\alpha\ \tau}{k}$$

$$C_2 = (\mathbf{x}_0)_y$$

d)

Sketch the velocity field for

$\alpha = k = 1$ and $t = 0$ and $t = 1$

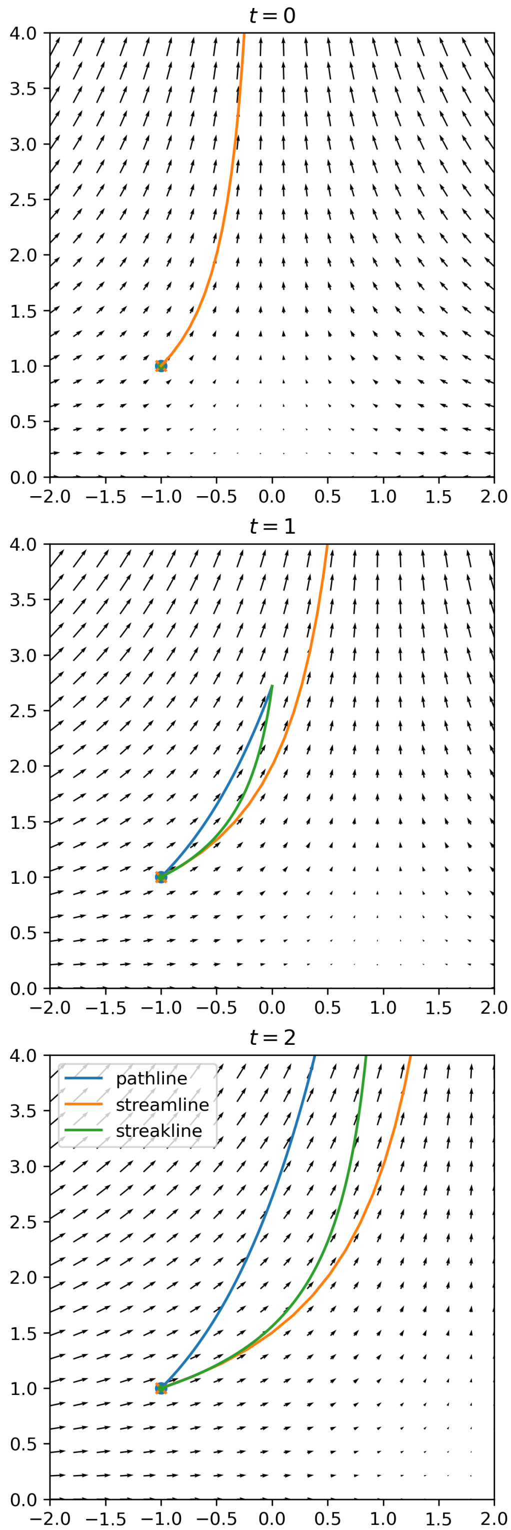
Mark a particular position $\mathbf{x}_0$ and draw

- one pathline
- one streamline
- one streakline starting at $t = 0$

for the chosen $\mathbf{x}_0$. The end positions of the lines should be within the plot.

partial solution

```
import numpy as np
import matplotlib.pyplot as plt


k = 1.
alpha = 1.

def U(t, x):
    u = -k*x[0] + alpha * t
    v = k * x[1]
    return u, v

def pathline(t, x0):
    C = x0[0] + alpha / k**2
    C2 = x0[1]
    T = np.linspace(0, t, 100)
    x = C * np.exp(-k*T) - alpha / k**2 + alpha * T / k
    y = C2 * np.exp(k*T)
    return [x, y]

def streamline(tstar, x0):
    C = x0[0] - alpha * tstar / k
    C2 = x0[1]
    s = np.linspace(0, 10, 100)
    x = C * np.exp(-k*s) + alpha * tstar/k
    y = C2 * np.exp(k*s)
    return [x, y]

def streakline(t, x0):
    tau = np.linspace(0, t, 100)
    theta = t - tau
    C = x0[0] - alpha * tau / k + alpha / k**2
    C2 = x0[1]
    x = C * np.exp(-k*theta) + alpha * (tau + theta)/k - alpha/k**2
    y = C2 * np.exp(k*theta)
    return [x, y]


xv, yv = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(0, 4, 20), indexing='xy')

fig, ax = plt.subplots(3, 1, layout='constrained', figsize=(4, 12))

x0 = [-1., 1]
time = 0
Ux, Uy = U(time, [xv, yv])
PLx, PLy = pathline(time, x0)
STx, STy = streamline(time, x0)
SKx, SKy = streakline(time, x0)
ax[0].quiver(xv, yv, Ux, Uy)
ax[0].plot(PLx, PLy, label='pathline')
ax[0].scatter(x0[0], x0[1], marker='o')
ax[0].plot(STx, STy, label='streamline')
ax[0].scatter(x0[0], x0[1], marker='x')
```


$t = 0$
$t = 1$
$t = 2$
pathline
streamline
streakline
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0