Do not open the exam sheet until prompted!

Continuum Mechanical Modeling for Simulation Science
Exam | WS24/25 | 18.03.2025

Permitted Aids:

- Document-proof writing utensils, but no red pen.
- The cheat sheet as provided by the organizer of the exam, either self-printed or in form of a copy provided during the exam.
- Other aids, especially the use of a calculator, are not allowed.

Hints:

- The use of cell phones during the exam is considered as an attempt to cheat.
- Any modification of the cheat sheet prior to the exam leads to exclusion of the exam.
- You have a total of **120 minutes** to complete the exam.
- To pass the exam, **50%** of the possible points are sufficient.
- Please begin each assignment on the sheet on which the assignment is formulated. If you use any of the attached blank sheets in addition to the blank page opposite, please indicate "Continue on another sheet" on the first sheet. *Please mark each sheet with your name and matriculation number — even the blank sheets used.*
- By signing this form, you assure that, to the best of your knowledge, you are fit to take the exam at the beginning of the exam and that the exam performance was performed by you without any unauthorized aids.

Matriculation number: ____ ____ ____ ____ ____ ____

Last name, first name: ______________________________

Signature: ______________________________

1. Incompressible Navier-Stokes Fourier Boussinesq

You have given the incompressible Navier-Stokes-Fourier system in Boussinesq formulation:

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho_0} \nabla \left(p - \rho_0 g z \right) + \nu \triangle \mathbf{v} - \mathbf{g} B (T - T_0) \\
\partial_t \left(\rho_0 c_p T \right) + \nabla \cdot \left(\rho_0 c_p T \mathbf{v} \right) &= \nabla \cdot (\kappa \nabla T) + \mathbf{S}.
\end{aligned}
$$

where ρ_0, ν, B, T_0, c_p, κ and $\mathbf{g} = g\mathbf{e}_z$ are constants of the system. $\mathbf{e}_z$ is the unit vector in z-direction.

1. Name all unknowns and parameters, including their SI units.

2. By introducing the following scales:

- Length scale L,
- Velocity scale V,
- Time scale $\frac{L}{V}$, and
- Pressure scale $\mu \frac{V}{L}$,

re-write the momentum equation in a dimensionless formulation, such that the viscosity term is scaled to one. Stay with the vector notation for convenience.

Which dimensionless numbers remain in the system?

3. Derive the dimensionless temperature equation. Note, that you can make use of the Peclet number Pe, which is defined as $\mathrm{Pe} = L\frac{V}{\alpha}$, with α being the thermal diffusivity.

4. The Prandtl number (Pr) denotes the ratio between momentum diffusivity ν and the thermal diffusivity α. Write the dimensionless temperature equation in terms of the Prandtl number.

Tip

First derive an expression of the Pr number in terms of Re and Pe

Points: 10

2. Heat Equation

Let's assume we have given a heat equation of constant advection velocity

$$\underbrace{\frac{\partial}{\partial t}T}_{\text{(I)}} - \underbrace{v\frac{\partial}{\partial x}T}_{\text{(II)}} = \underbrace{\alpha\frac{\partial^2}{\partial x^2}T}_{\text{(III)}} \tag{1}$$

on a domain $x = [0, \infty)$ with boundary conditions

$$T\big|_{x=0} = T_{\text{wall}}, \qquad T\big|_{x\to\infty} = T_{\text{inf}} = \text{const.}, \qquad T_{\text{wall}} > T_{\text{inf}} \tag{2}$$

and initial condition $T(x, 0) = T_{\text{inf}}$.

We introduce the dimensionless variables

$$t := t_0\tilde{t}, \qquad x := x_0\tilde{x}, \qquad v := v_0\tilde{v}, \text{and } T := \underbrace{(T_{\text{wall}} - T_{\text{inf}})}_{T_0}\tilde{T} + T_{\text{inf}}.$$

1. Explain the physical meaning of the terms (I), (II), and (III) of the heat equation.
2. What is α called? Give its SI units
3. Write both equation Equation 1 and the boundary conditions Equation 2 in terms of the dimensionless variables.
4. What is the physical meaning of the Peclet number Pe?

Now consider a physical regime characterized by $t_0 \gg \frac{x_0^2}{\alpha}$.

5. How does the dimensionless equation derived in Task 3. read in this physical regime?
6. Derive a solution for the temperature profile $\tilde{T}$ in terms of $Pe, \tilde{v}$ and $\tilde{x}$
7. If the α halves, how does the velocity need to change to retain the same profile?

Points: 10

3. Shallow Water Equations

The following equation describe the incompressible Navier-Stokes equations in dimensionless form using a scaling with dimensionless numbers in (ϵ, Fr). The system uses a rotated coordinate system with inclination angle ζ:

$$\begin{aligned} &\partial_x u + \partial_z w = 0 \\ &\epsilon Fr^2 \left(\partial_t u + \partial_x u^2 + \partial_z (uw)\right) = \epsilon \partial_x \sigma_{xx} + \partial_z \sigma_{xz} + \sin(\zeta) \\ &\epsilon^2 Fr^2 \left(\partial_t w + \partial_x (uw) + \partial_z (w^2)\right) = \epsilon \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \cos(\zeta) \,. \end{aligned} \tag{3}$$

We use a material model of the form:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix} = -pI + \begin{pmatrix} 0 & \tau_{xz} \\ \tau_{xz} & 0 \end{pmatrix} \,. \tag{4}$$

In the following, we denote $b(x)$ as the bottom topography, $s(t,x)$ as the free surface and $h(t,x) = s(t,x) - b(x)$ as the water height.

Assume the shear stress τ_{xz} is given by

$$\tau_{xz} = -\epsilon^{-1} \nu \partial_z u - \epsilon^0 \sin(\zeta)(z - b) \tag{5}$$

with a linear velocity profile

$$u(t,x,z) = \hat{u}(t,x) \frac{z - b(x)}{h(t,x)} \tag{6}$$

where $\hat{u}(t,x) = u(t,x,z = s(t,x))$ denotes the velocity at the free-surface.

1. Derive an expression for the shear stress τ_{xz} (Equation 5) using the velocity profile (Equation 6).
2. Substitute your result from task 1. and the material model (Equation 4) into Equation 3. and simplify the resulting system.
3. Now we consider the ansatz

$$\begin{aligned} u &= \epsilon^0 u^{(0)} + \epsilon^1 u^{(1)} + \mathcal{O}(\epsilon^2) \\ w &= \epsilon^0 w^{(0)} + \epsilon^1 w^{(1)} + \mathcal{O}(\epsilon^2) \\ p &= \epsilon^0 p^{(0)} + \epsilon^1 p^{(1)} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Substitute it into the expression of your simplified system (task 2.) and neglect terms of order ϵ^2 or higher.

4. Write down the asymptotic expansion by separating the system according to its scales. Your result should yield non-trivial equations on the scales (ϵ^0) and (ϵ^1).

The pressure distribution now takes the form

$$p(t,x,z) = A(t,x)(h - (z - b)) \tag{7}$$

5. Compute $A(t,x)$ by first deriving equations for $p^{(0)}$ and $p^{(1)}$ and assuming a stress-free free-surface boundary condition $p(t,x,z = s) = 0$.

6. Is the pressure hydrostatic? Explain your answer.

Points: 10

4. Freezing body of water

Let's assume we have a large box ($x \in (-\infty, 0]$) filled with ice water at $T_0 = T_m = 0°\text{C}$.

We control the temperature at the right wall of the box to be constant: $T_w(t) = T(t, x = 0) = -10°\text{C}$.

1. Sketch the experimental situation and draw a corresponding temperature profile (T over x plot) for time $t_0 = 0$ and some time $t_1 > 0$.
2. Process model:
    1. Write down a mathematical model (PDE) to describe the temperature evolution in each domain.
    2. Which condition is needed to close the system? Simplify the condition based on our experimental situation.
    3. State the unknowns of the resulting closed PDE system.
3. Analytical solution:
    1. What is the convenient (technical, mathematical) effect of introducing the similarity variable $\zeta = \frac{x}{\sqrt{t}}$?
    2. Rewrite the temperature evolution in terms of the similarity variable ζ.

From now on, assume the following Ansatz for the propagation of the interface:

$$X_m(t) = 2\sqrt{\alpha t}\lambda$$

with λ constant and α the thermal diffusivity.

4. Show that

$$T(t, x) = T_w - (T_w - T_m)\frac{erf\left(\frac{x}{2\sqrt{\alpha t}}\right)}{erf\left(\frac{X_m(t)}{2\sqrt{\alpha t}}\right)}$$

is a solution for one domain of your system.

Points: 10

5. Short Problems

5.1 Index Notation and Component Form

In the Langrangian description, using vectorial notation the mass balance law is given by

$$\frac{D\rho}{Dt} = 0$$

and the momentum balance law is given by

$$\frac{D\mathbf{v}}{Dt} = \frac{1}{\rho}\nabla\boldsymbol{\sigma} + b$$

1. Write down the mass balance in the Eulerian description using index notation
2. Give the component form of the momentum balance in the Eulerian description

Points: 4

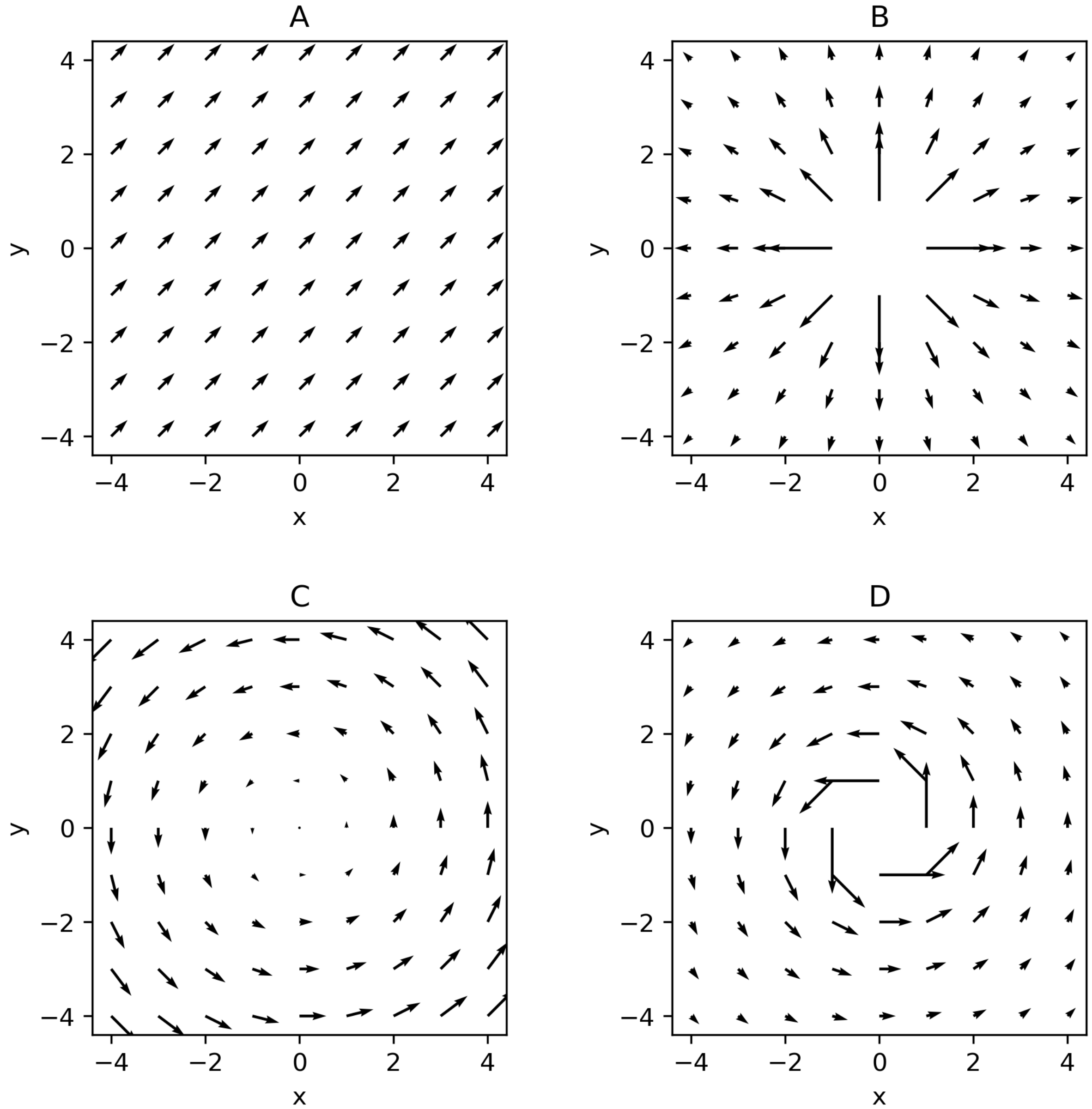
5.2 Velocity decomposition

We infer on an expression for the flow field given by

$$v(x) = \begin{pmatrix} 1-y \\ 1+x \\ 0 \end{pmatrix} \tag{8}$$

1. Decompose the given velocity field by calculating the axial vector of spin tensor **W** and the strain rate tensor **D**
2. Superposition of which **two** of the following flow fields would lead to the flow field as described in Equation 8.

☐ A	☐ B	☐ C	☐ D



Points: 4

5.3 Eulerian and Lagrangian frame of reference

Given the trajectory $\mathbf{X}(t, \mathbf{x}_0) = (X(t, \mathbf{x}_0), Y(t, \mathbf{x}_0), Z(t, \mathbf{x}_0))^T = (X, Y, Z)^T$

$$
\begin{aligned}
X(t, \mathbf{x}_0) &= x_0 \cos(t) + y_0 \sin(t) \\
Y(t, \mathbf{x}_0) &= -x_0 \sin(t) + y_0 \cos(t) \\
Z(t, \mathbf{x}_0) &= 0
\end{aligned}
$$

with $\mathbf{x}_0 = (x_0, y_0, z_0)^T$ indicating coordinates at $t = 0$.

a)

Given the scalar field

$$\phi(t, \mathbf{X}) = x_0^2 + y_0^2$$

Which of the following expressions describe ϕ in the Eulerian frame:

- ☐ $\phi(t, \mathbf{x}) = x^2 + y^2$
- ☐ $\phi(t, \mathbf{x}) = X^2 + Y^2$
- ☐ $\phi(t, \mathbf{x}) = x_0^2 + y_0^2$
- ☐ $\phi(t, \mathbf{X}) = x^2 + y^2$
- ☐ $\phi(t, \mathbf{X}) = X^2 + Y^2$
- ☐ $\phi(t, \mathbf{X}) = x_0^2 + y_0^2$

b)

Given the scalar field $\psi(t, \mathbf{X}(t)) = X^2 + Y^2$. Which of the following expressions describe ψ in the Eulerian frame:

- ☐ $\psi(t, \mathbf{x}) = x^2 + y^2$
- ☐ $\psi(t, \mathbf{x}) = X^2 + Y^2$
- ☐ $\psi(t, \mathbf{x}) = x_0^2 + y_0^2$
- ☐ $\psi(t, \mathbf{X}) = x^2 + y^2$
- ☐ $\psi(t, \mathbf{X}) = X^2 + Y^2$
- ☐ $\psi(t, \mathbf{X}) = x_0^2 + y_0^2$

Points: 4

5.4 Streamlines, Pathlines, Streaklines

Consider the velocity field

$$\mathbf{v}(t, x, y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -kx + \alpha t \\ ky \end{pmatrix}$$

with k, α being positive constants.

Decide which problem statement belongs to the computation of streamlines, pathlines or streaklines respectively.

> **Note**
>
> This question grants 4 points if all subtasks are answered correctly and 0 points if there is any mistake.

a)

We introduce the **local time** θ and the start time τ. We solve the following ODE

$$\frac{d\mathbf{X}(\theta;\tau)}{d\theta} = \mathbf{v}(t = \theta + \tau, \mathbf{X}(\theta;\tau)) = \begin{pmatrix} -k\ \mathbf{X}_x(\theta;\tau) + \alpha\ (\theta + \tau) \\ k\ \mathbf{X}_y(\theta;\tau) \end{pmatrix}$$

with initial conditions $\mathbf{X}(\theta = 0) = \mathbf{x}_0$.

b)

We solve the following ODE

$$\frac{d\mathbf{X}}{dt} = \mathbf{v} = \begin{pmatrix} -k\ \mathbf{X}_x + \alpha\ t \\ k\ \mathbf{X}_y \end{pmatrix}$$

with initial conditions $\mathbf{X}(t = 0, \mathbf{x}_0) = \mathbf{x}_0$.

c)

We fix time $t = t^*$. Then we solve the following ODE

$$\frac{d\mathbf{X}(s;\mathbf{x}_0)}{ds} = \mathbf{v}(t^*) = \begin{pmatrix} -k\ \mathbf{X}_x + \alpha\ t^* \\ k\ \mathbf{X}_y \end{pmatrix}$$

with initial conditions $\mathbf{X}(s = 0) = \mathbf{x}_0$.

Your answer

Streamline:	
Streakline:	
Pathline:	

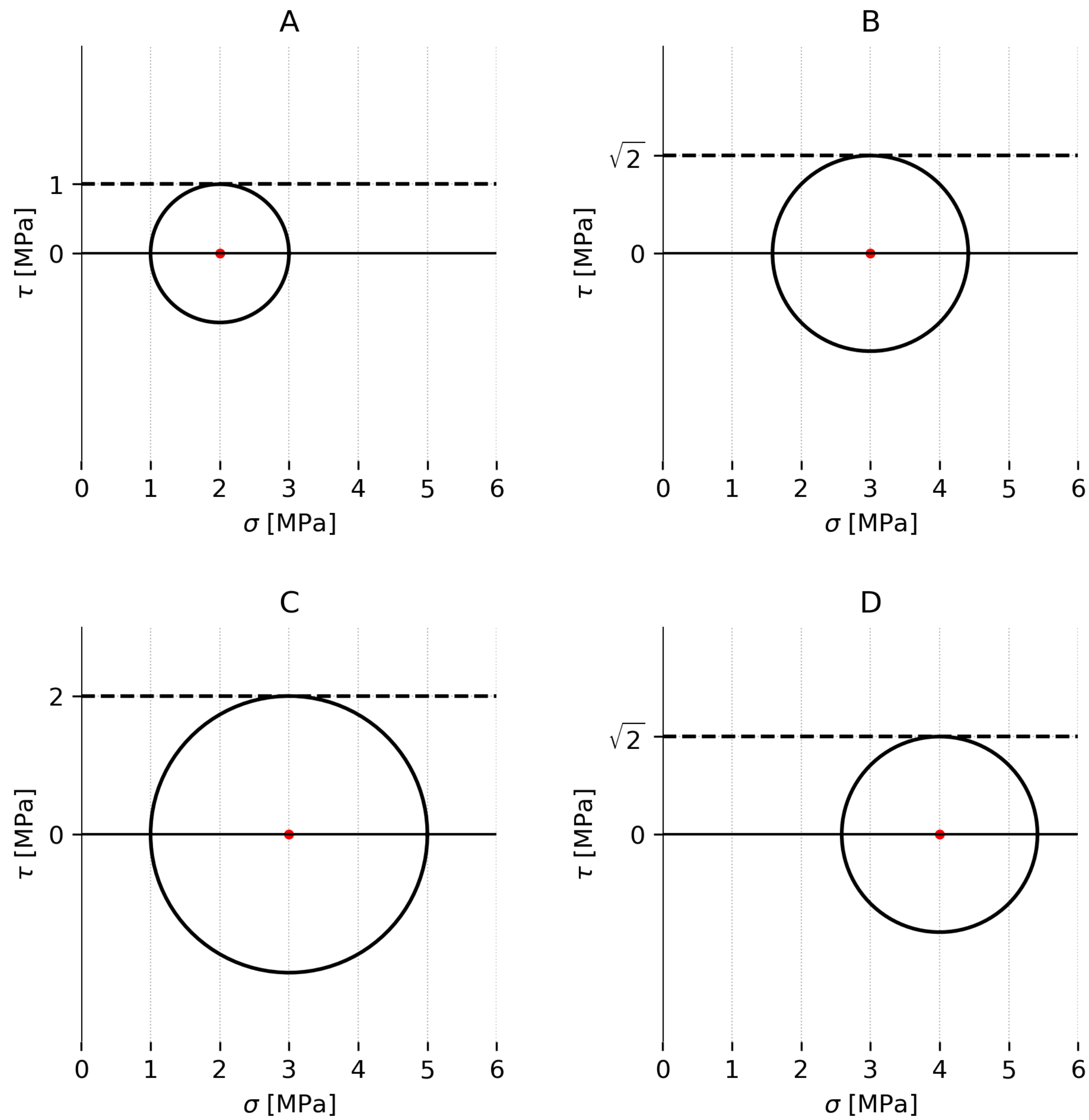
Points: 4

5.5 Mohr's circle

The stress state at a point in a 2-dimensional continuum is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \text{MPa}$$

1. Identify which of the following *Mohr* circle diagram corresponds to the the stress tensor $\boldsymbol{\sigma}$



2. For the stress tensor $\boldsymbol{\sigma}$

   a. Compute the principal stresses σ_1 and σ_2
   b. Compute the maximum shear stress $\tau_{\max}$

Points: 4

5.6 Stress tensor

1. The Cauchy stress $\boldsymbol{\sigma}$ at any point $\mathbf{x}$ in a continuous body can be decomposed into spherical and deviatoric part: $\boldsymbol{\sigma} = \mathbf{S}_s + \mathbf{S}_d$, wherein

$\mathbf{S}_s = -p\mathbf{I}$ describes:	
$\mathbf{S}_d = \boldsymbol{\sigma} - \mathbf{S}_s$ describes:	

Fill in the table above with the correct options from below.

A. part of stress that tends to change the volume without changing the shape

B. part of stress that tends to change the shape without changing the volume

2. Which of the following implies the symmetry of the stress tensor? ______________

   A. Conservation of angular momentum

   B. Conservation of translational momentum

   C. Conservation of energy

Points: 4

5.6 Limits of a simple viscoelastic material

A simple model to determine the deformation of a viscoelastic material is given by

$$\rho \frac{\partial^2}{\partial t^2} u = \alpha \frac{\partial^2}{\partial x^2} u + \beta \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} u \quad , \tag{9}$$

with constants $\rho, \alpha, \beta \in \mathbb{R}$ and unknown displacement field $u = u(t,x) \in \mathbb{R}$. We want to study solutions of this equation that obey the form

$$u(t,x) = A\, exp(i(x - c\, t)) \tag{10}$$

with $i^2 = -1$ and $A, c \in \mathbb{R}$.

> Tip
>
> Recall Euler's formula $exp(i\,\theta) = cos(\theta) + i\, sin(\theta)$
> A wave solution is **damped** if $u \to 0$ for $t \to \infty$.

> Note
>
> Multiple or no answers may be correct.

a)

The **elastic limit** is given for

- ☐ $\alpha > 0$ and $\beta = 0$
- ☐ $\alpha = 0$ and $\beta > 0$

b)

The **viscous limit** admits

- ☐ stationary solutions
- ☐ transient solutions
- ☐ solutions that are damped exponentially
- ☐ solutions that are not damped at all
- ☐ unphysical solutions with exponential growth

c)

The **elastic limit** admits:

- ☐ stationary solutions
- ☐ transient solutions
- ☐ solutions that are damped exponentially
- ☐ solutions that are not damped at all
- ☐ unphysical solutions with exponential growth

d)

For $\alpha > 0$ and $\beta > 0$, we may observe

- ☐ stationary solutions
- ☐ transient solutions
- ☐ solutions that are damped exponentially
- ☐ solutions that are not damped at all
- ☐ unphysical solutions with exponential growth

Points: 4

5.7 Perturbation from a lake at rest

The shallow water equations with velocity profile correction in one space dimension are given by:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

where $h(x,t)$ describes the fluid depth, $u(x,t)$ the mean flow velocity and $g = 9.81\mathrm{m/s}^2$ is the gravitational acceleration.

Assume a shallow lake, currently at rest. That means $h = H$ and $u = 0$. A stone is thrown into the lake and causes a small surface perturbation in height and velocity, denoted by h_1 and u_1.

The linearization can be performed by looking at small perturbations from a ground state (h_0, u_0), e.g. by defining: $h(x,t) = h_0(x,t) + \delta\, h_1(x,t) + \mathcal{O}(\delta^2)$ and $u(x,t) = u_0(x,t) + \delta\, u_1(x,t) + \mathcal{O}(\delta^2)$, where δ denotes the small amplitude of the perturbation.

1. Decide which system of equations describe the correct linearization of Equation 11.

☐	$\frac{\partial}{\partial t}\begin{pmatrix} H \\ Hu_1 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} Hu_1 \\ Hu_1^2 + g\frac{H^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
☐	$\frac{\partial}{\partial t}\begin{pmatrix} \delta h_1 \\ \delta h_1 u_1 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \delta h_1 u_1 \\ \delta h_1 u_1^2 + \delta g\frac{h_1^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
☐	$\frac{\partial}{\partial t}\begin{pmatrix} \delta h_1 \\ \delta H u_1 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \delta H u_1 \\ \delta^2 H u_1^2 + g\frac{(H+\delta h_1)^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
☐	$\frac{\partial}{\partial t}\begin{pmatrix} \delta h_1 \\ \delta H u_1 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \delta H u_1 \\ g\delta H h_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2. The linearized system can by analyzed by means of a wave ansatz given by $h_1(x,t) = A\, e^{i(k\,x - \omega\, t)}$. State what A, i, k and ω stand for?

A
i
k
ω

5.8 Shape factor

The shallow water equations with velocity profile correction in one space dimension are given by:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ \alpha hu^2 + g\frac{h^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $h(x,t)$ describes the fluid depth, $u(x,t)$ the mean flow velocity and $g = 9.81\text{m/s}^2$ is the gravitational acceleration.

Mark the correct shape factor α for a velocity profile defined by

$$u(y) = \begin{cases} 2u_s y/h & y \leq 0.5h \\ u_s & y > 0.5h \end{cases}$$

☐	32/27
☐	1.2
☐	64/45
☐	4/3

Points: 4

5.9 Darcy's law

Darcy's law states

$$\mathbf{q} = -\frac{\kappa}{\mu}\left(\nabla p - \rho \mathbf{g}\right) .$$

1. What does $\mathbf{q}, \kappa, \mu, \mathbf{g}$ stand for?
2. State the SI-units of $\mathbf{q}, \kappa, \mu, \mathbf{g}$?

$\mathbf{q}$:

κ:

μ:

$\mathbf{g}$:

Points: 4

5.10 Bloch-Torrey model

Given the simplified Bloch-Torrey model:

$$
\begin{aligned}
\frac{\partial \mathbf{M}}{\partial t}(\mathbf{x}, t) = & - \underbrace{\mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{M}(\mathbf{x}, t)}_{(A)} + \underbrace{\gamma\, \mathbf{M}(\mathbf{x}, t) \times \mathbf{B}}_{(B)} \\
& \underbrace{- \quad \cdot \left[\begin{pmatrix} \alpha(\mathbf{x}) & 0 & 0 \\ 0 & \alpha(\mathbf{x}) & 0 \\ 0 & 0 & \alpha(\mathbf{x}) \end{pmatrix} \nabla \mathbf{M} \right]}_{(C)} \\
& \underbrace{+ \begin{pmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{pmatrix} (\mathbf{M} - \mathbf{M}_0)}_{(D)}
\end{aligned}
$$

Use labels (A), (B), (C) and (D) to identify (if appropriate):

Excitation term:

Relaxation term:

Advection term:

Diffusion term:

Points: 4