Do not open the exam sheet until prompted!

Continuum Mechanical Modeling for Simulation Science
Exam | SS24 | 14.08.2024

Permitted Aids:

- Document-proof writing utensils, but no red pen.
- The cheat sheet as provided by the organizer of the exam, either self-printed or in form of a copy provided during the exam.
- Other aids, especially the use of a calculator, are not allowed.

Hints:

- The use of cell phones during the exam is considered as an attempt to cheat.
- Any modification of the cheat sheet prior to the exam leads to exclusion of the exam.
- You have a total of **120 minutes** to complete the exam.
- To pass the exam, **50%** of the possible points are sufficient.
- Please begin each assignment on the sheet on which the assignment is formulated. If you use any of the attached blank sheets in addition to the blank page opposite, please indicate "Continue on another sheet" on the first sheet. *Please mark each sheet with your name and matriculation number — even the blank sheets used.*
- By signing this form, you assure that, to the best of your knowledge, you are fit to take the exam at the beginning of the exam and that the exam performance was performed by you without any unauthorized aids.

Matriculation number: ____ ____ ____ ____ ____ ____

Last name, first name: ______________________________

Signature: ______________________________

1. Incompressible Navier-Stokes Equations

Consider the incompressible Navier-Stokes equations in 3D

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{b}. \end{aligned} \tag{1}$$

1. What does ν stand for? Write down the SI units for ν.
2. Given that $\rho, \nu, \mathbf{b}$ are known, how many individual equations and unknowns (variable name and number of component) does the system of equations in Equation 1 comprise of? Identify the unknowns.
3. In the momentum balance equation in Equation 1 identify
   1. Convective terms
   2. Diffusive terms
   3. Source terms

In order to derive the dimensionless formulation of Equation 1, we define the following scaled variables:

$$x_i = L_0 \tilde{x}_i, \quad v_i = V_0 \tilde{v}_i, \quad \mathbf{b} = g\tilde{\mathbf{b}}, \quad t = T_0 \tilde{t} = \frac{L_0}{V_0} \tilde{t} \quad \text{and} \quad p = \rho {V_0}^2 \tilde{p}$$

4. Derive the dimensionless mass balance from Equation 1. Describe how the form differs from the original form of the mass balance equation.
5. Derive the dimensionless momentum balance in terms of the Reynolds number Re and Froude number Fr.
6. With the introduction of a new pressure scaling $p = \frac{\nu \rho V_0}{L_0} \hat{p}$, derive the dimensionless form of the momentum balance.
7. Based on the dimensionless forms of the momentum balance, identify which part of the mathematical model will vanish and name the resulting process model when:
   1. $Re >> 1$ for $\tilde{p}$
   2. $Re << 1$ for $\hat{p}$

Points: 15

2. Heat Equation

Let's assume we have given a heat equation of constant advection speed

$$\underbrace{\frac{\partial}{\partial t} T}_{\text{(I)}} - \underbrace{v \frac{\partial}{\partial x} T}_{\text{(II)}} = \underbrace{\lambda \frac{\partial^2}{\partial x^2} T}_{\text{(III)}} \tag{2}$$

on a domain $x = [0, \infty)$ with boundary conditions

$$T\big|_{x=0} = T_{\text{wall}}, \qquad T\big|_{x \to \infty} = T_{\text{inf}} = \text{const.}, \qquad T_{\text{wall}} > T_{\text{inf}} \tag{3}$$

and initial condition $T(x,0) = T_{\text{inf}}$.

1. Explain the physical meaning of the terms (I), (II), and (III) of the heat equation.
2. What is λ called? Give its SI units.

We introduce the dimensionless variables

$$t := t_0\tilde{t}, \qquad x := x_0\tilde{x}, \qquad v := v_0\tilde{v}, \text{and } T := \underbrace{(T_{\text{wall}} - T_{\text{inf}})}_{T_0}\tilde{T} + T_{\text{inf}}.$$

and arrive at the dimensionless form of the heat equation

$$\frac{{x_0}^2}{\lambda t_0}\frac{\partial}{\partial \tilde{t}}\tilde{T} - Pe\tilde{v}\frac{\partial}{\partial \tilde{x}}\tilde{T} = \frac{\partial^2}{\partial^2 \tilde{x}}\tilde{T}$$

Now consider a physical regime characterized by $t_0 \gg \frac{x_0^2}{\lambda}$.

3. How does the equation read in this physical regime?
4. What is the physical meaning of the Peclet number Pe?
5. Show that the temperature profile $\tilde{T} = \exp(-Pe\tilde{v}\tilde{x})$ is a solution to the equation arising in 3.
6. Indicate through a sketch, how the temperature profile changes as the velocity increases as the velocity increases via a qualitative plot at two times. Note, that the plot does not have to be quantitatively correct.
7. If the velocity doubles, how does λ need to change to retain the same profile?

Points: 15

3. Shallow Water Equations

The incompressible Navier-Stokes equations in dimensionless form using a scaling in (ϵ, Fr) in a rotated coordinate system inclined at angle ζ are given by:

$$\begin{aligned}
&\partial_x u + \partial_z w = 0 \\
&\epsilon Fr^2 \left(\partial_t u + \partial_x u^2 + \partial_z (uw)\right) = \epsilon \partial_x \sigma_{xx} + \partial_z \sigma_{xz} + sin(\zeta) \\
&\epsilon^2 Fr^2 \left(\partial_t w + \partial_x (uw) + \partial_z (w^2)\right) = \epsilon \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - cos(\zeta)\ .
\end{aligned} \tag{4}$$

We use a material model of the form:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix} = -pI + \begin{pmatrix} 0 & \tau_{xz} \\ \tau_{xz} & 0 \end{pmatrix}\ .$$

In the following, we denote $b(x)$ as the bottom topography, $s(t,x)$ as the free surface and $h(t,x) = s(t,x) - b(x)$ as the water height.

Assume the shear stress τ_{xz} is given by

$$\tau_{xz} = -\epsilon^{-1}\nu\partial_z u - \epsilon^0 sin(\zeta)(z-b)$$

with a linear velocity profile

$$u(t,x,z) = \hat{u}(t,x)\frac{z-b(x)}{h(t,x)} \tag{5}$$

where $\hat{u}(t,x) = u(t,x,z=s(t,x))$ denotes the velocity at the free-surface.

1. Derive an expression for the shear stress τ_{xz} using the assumption on the velocity profile (Equation 5).
2. Derive the asymptotic expansion of Equation 4 using

$$\begin{aligned} u &= \epsilon^0 u^{(0)} + \epsilon^1 u^{(1)} + \mathcal{O}(\epsilon^2) \\ w &= \epsilon^0 w^{(0)} + \epsilon^1 w^{(1)} + \mathcal{O}(\epsilon^2) \\ p &= \epsilon^0 p^{(0)} + \epsilon^1 p^{(1)} + \mathcal{O}(\epsilon^2) \end{aligned}$$

Now we neglect terms of order ϵ^2 or higher. The pressure distribution now takes the form $p(t,x,z)$

$$p(t,x,z) = A(t,x)(h-(z-b))$$

where $A(t,x)$ is a function of time t and dimension x.

3. Compute the pressure distribution $p(t,x,z)$ by first deriving equations for $p^{(0)}$ and $p^{(1)}$ and assuming a stress-free free-surface boundary condition $p(t,x,z=s)=0$.
4. Is the pressure hydrostatic? Explain your answer.
5. Based on the asymptotic expansion, compute the depth averaged equations of Equation 4. Use

$$\begin{aligned} \int_b^s \partial_x p(t,x,z)\,dz =& \partial_x \left(\int_b^s p(t,x,z)\,dz \right) \\ & - \left(p(t,x,z)\Big|_{z=s} \partial_x s - p(t,x,z)\Big|_{z=b} \partial_x b \right) \end{aligned} \tag{6}$$

and

$$\int_b^s \partial_t u + \partial_x(u^2) + \partial_z(uw)\,dz = \partial_t(h\bar{u}) + \partial_x(\alpha h \bar{u}^2) + R \tag{7}$$

for the left hand side of the momentum balance. $\bar{u}$ denotes the mean velocity and R a residual term.

Points: 15

4. Freezing body of water

Let's assume we have a large box ($x \in (-\infty, 0]$) filled with ice water at $T_0 = T_m = 0°C$.

We control the temperature at the right wall of the box to be constant: $T_w(t) = T(t, x=0) = -10°C$.

1. Sketch the experimental situation and draw a corresponding temperature profile (T over x plot) for time $t_0 = 0$ and some time $t_1 > 0$.
2. Process model:
    1. Write down a mathematical model (PDE) to describe the temperature evolution in each domain.

2. Which condition is needed to close the system? Simplify the condition based on our experimental situation.
3. State the unknowns of the resulting closed PDE system.

3. Analytical solution:

1. What is the convenient (technical, mathematical) effect of introducing the similarity variable $\zeta = \frac{x}{\sqrt{t}}$?
2. Rewrite the temperature evolution in terms of the similarity variable ζ.

From now on, assume the following Ansatz for the propagation of the interface:

$$X_m(t) = 2\sqrt{\alpha t}\lambda$$

with λ constant and α the thermal diffusivity.

4. Show that

$$T(t,x) = T_w - (T_w - T_m)\frac{erf\left(\frac{x}{2\sqrt{\alpha t}}\right)}{erf\left(\frac{X_m(t)}{2\sqrt{\alpha t}}\right)}$$

is a solution for one domain of your system.

5. Derive a homogeneous function ($F(\lambda) = 0$) that could be solved numerically to determine the interface position $X_m(t)$ by substituting the previous solution into the closure (task 2.2) and simplify.

Points: 15

5. Small Problems

5.1 Linear elasticity

During a deformation experiment on a sample of an unknown material, the following deformation gradient was observed:

$$\mathbf{F} = \begin{bmatrix} 2.000 & 0.000 & 0.000 \\ 0.000 & 1.0 & 0.500 \\ 0.000 & -0.500 & 1.0 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j$$

1. Compute the displacement gradient $\mathbf{H}$.

Tip

The deformation gradient $\mathbf{F} = \mathbf{H} + \mathbf{I}$

2. Compute the small strain tensor $\mathbf{D}$, which is given by the symmetric part of the displacement gradient.

3. Given the *Lame* parameters, $\lambda = 150 \times 10^3$MPa and $\mu = 75 \times 10^3$MPa, compute the Cauchy stress tensor $\boldsymbol{\sigma}$.

Points: 4

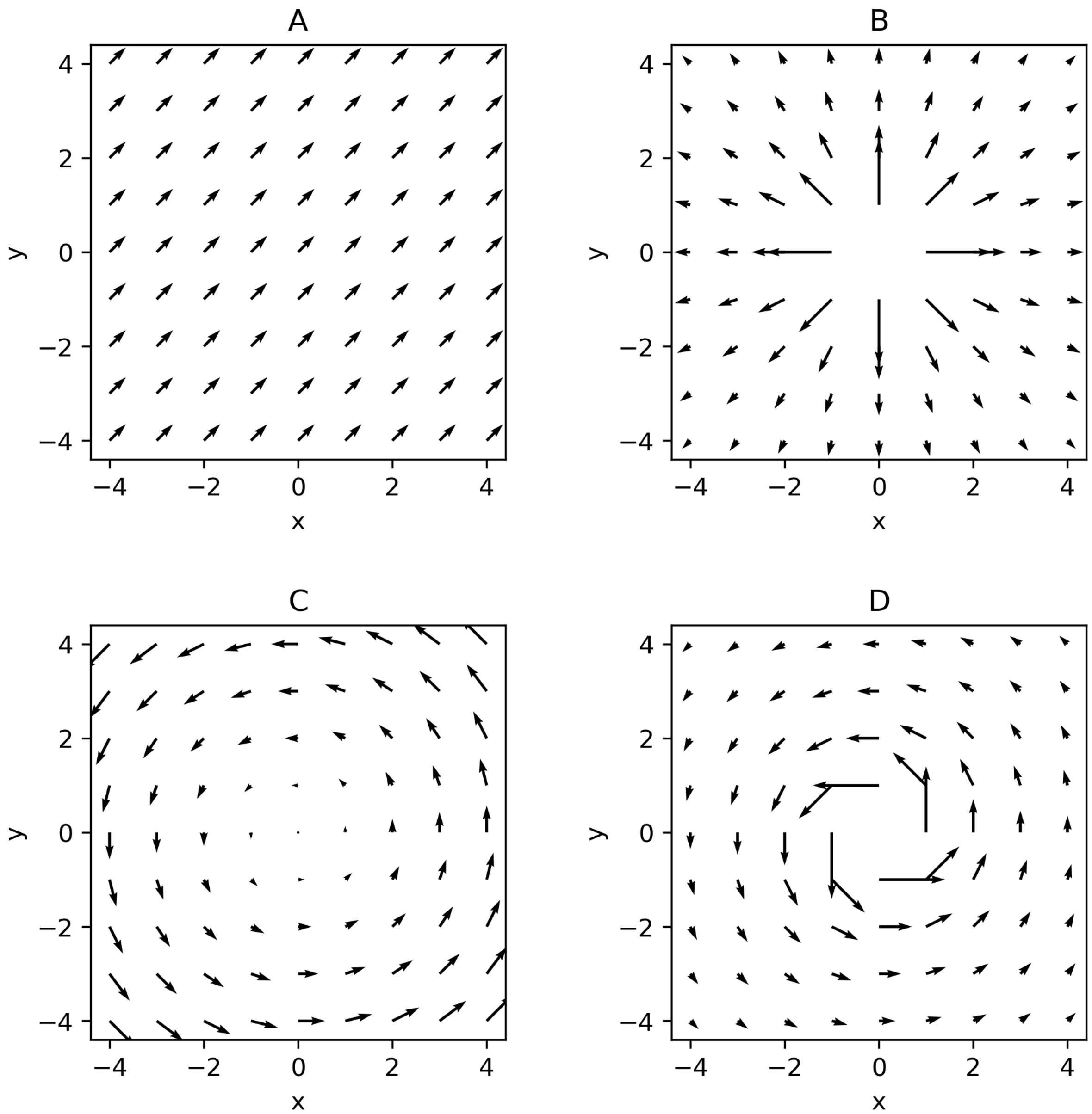
5.2 Velocity decomposition

We infer on an expression for the flow field given by

$$v(x) = \begin{pmatrix} 1-y \\ 1+x \\ 0 \end{pmatrix} \tag{8}$$

1. Decompose the given velocity field by calculating the axial vector of spin tensor **W** and the strain rate tensor **D**
2. Superposition of which **two** of the following flow fields would lead to the flow field as described in Equation 8.

☐ A	☐ B	☐ C	☐ D



Points: 4

5.3 Finite differences

Consider the Burgers equation in one spatial dimension

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial^2 x^2}$$

In order to solve this equation numerically, we use the finite difference method. Using explicit Euler integration in time, backward differences for first order spatial derivatives and central differences for second derivatives, we arrive at the following discretization:

$$\frac{\boxed{1} - u_i^n}{\Delta t} + u_i^n \frac{\boxed{2} - \boxed{3}}{\Delta x} = \nu \frac{\boxed{4} - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Associate the missing discretized values $u_{i+1}^n, u_{i-1}^n, u_i^n, u_i^{n+1}$ to the boxes $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}$.

box	term
$\boxed{1}$	
$\boxed{2}$	
$\boxed{3}$	
$\boxed{4}$	

Points: 4

5.4 Deformation gradient

How can the deformation gradient $\mathbf{F}$ be described for the following cases? Fill in the blanks with the correct options given below

Pure rigid body displacement:	
Pure rigid body rotation:	
Pure stretching in axial direction:	
Pure Shear:	

A. Symmetric Tensor

B. Skew-Symmetric Tensor

C. Identity Tensor

D. Diagonal Tensor

E. Rotation Tensor

Points: 4

5.5 Stress tensor

1. The Cauchy stress $\boldsymbol{\sigma}$ at any point $\mathbf{x}$ in a continuous body can be decomposed into spherical and deviatoric part: $\boldsymbol{\sigma} = \mathbf{S}_s + \mathbf{S}_d$, wherein

$\mathbf{S}_s = -p\mathbf{I}$ describes:	
$\mathbf{S}_d = \boldsymbol{\sigma} - \mathbf{S}_s$ describes:	

Fill in the table above with the correct options from below.

A. part of stress that tends to change the volume without changing the shape

B. part of stress that tends to change the shape without changing the volume

2. Which of the following implies the symmetry of the stress tensor? ______________

   A. Conservation of angular momentum

   B. Conservation of translational momentum

   C. Conservation of energy

Points: 4

5.6 Bernoulli equation

In order to derive the Bernoulli equations

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = const$$

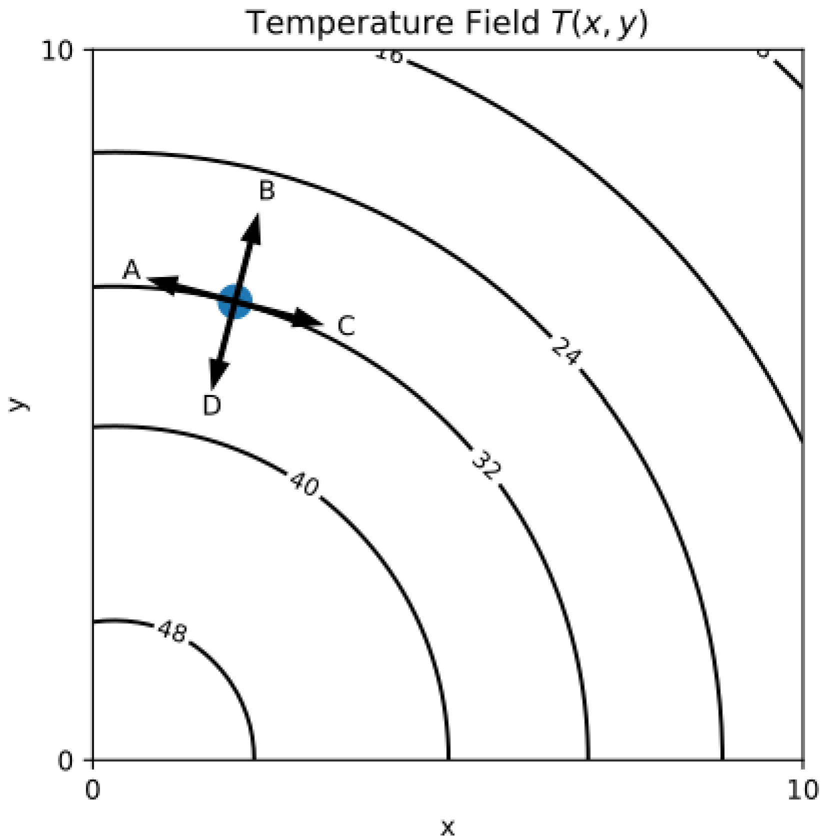
from the general mass and momentum balance, which of the following assumptions are used? Select **all** the correct options

- ☐ medium is in a stationary regime
- ☐ the fluid is isothermal
- ☐ body forces are given as a gravitational potential
- ☐ the medium is compressible
- ☐ the flow is turbulent
- ☐ we are given an incompressible fluid
- ☐ the flow is rotating
- ☐ we have a barotropic fluid
- ☐ the medium is exposed to hydrostatic pressure

Points: 4

5.7 Temperature field

The following plot of a temperature field is encountered during a simulation.



1. The temperature gradient ∇T at the blue dot has the direction given by _________.
2. The heat flux $\mathbf{q}$ at the blue dot has the direction given by _________.

Points: 4

5.8 Computational model development

A typical computational model development cycle involves both verification and validation. Describe in your own words the terms *verification* and *validation* as they have been used in this lecture.

Points: 4

5.9 Navier-Stokes-Boussinesq model

This is the Navier-Stokes-Fourier model with Boussinesq approximation:

$$\nabla \cdot \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0} \nabla (p - \rho_0 g z) + \nu \triangle \mathbf{v} - \mathbf{g} B (T - T_0)$$

$$\partial_t (\rho c_p T) + \nabla \cdot (\rho c_p T \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \mathbf{S}.$$

1. What is the physical meaning of κ?
2. What is the SI-units of κ?

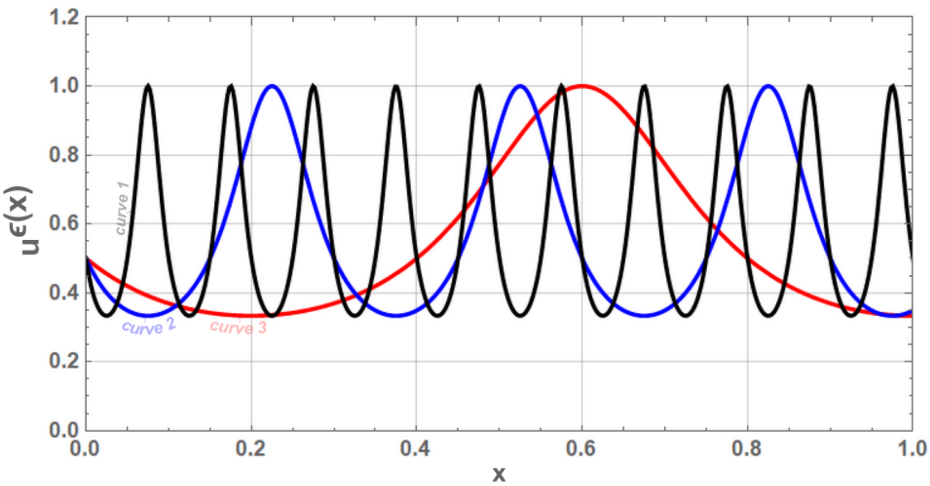
Points: 4

5.10 Homogenization

Given the PDE

$$\frac{d}{dx}\left(\frac{1}{1+2sin^2(\pi\frac{x}{\epsilon})}\frac{d}{dx}u^\epsilon(x)\right)=0 \qquad 0\le x\le L,$$

a direct finite-difference discretization yields the following result



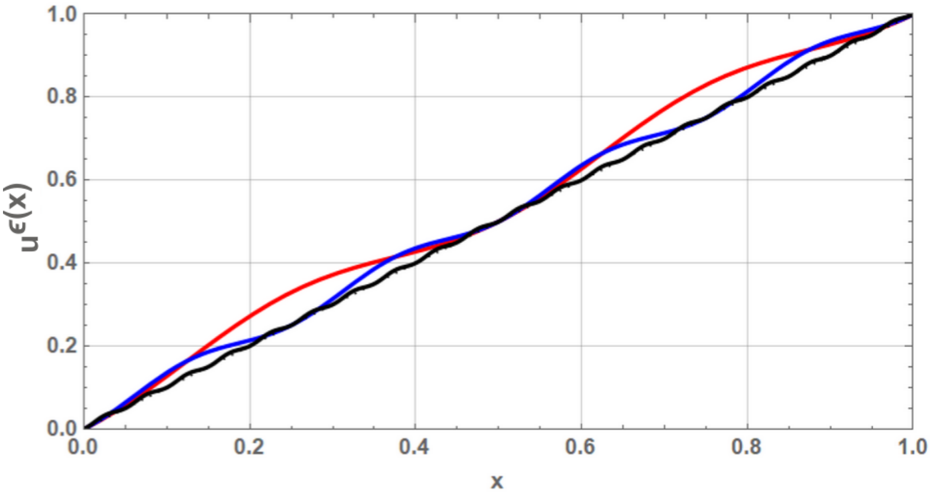
1. Indicate which line (red, blue, black) corresponds to which ϵ:

ϵ	your answer (color)
$\epsilon = 0.05$	
$\epsilon = 0.25$	
$\epsilon = 0.5$	

2. Why is the direct simulation approach from above not suited so solve the problem? (choose one)

☐ Instabilities due to oscillation
☐ No convergence for $\epsilon \to 0$
☐ A constant (unphysical) solution for $\epsilon \to \infty$

3. After homogenization, we obtained the following result



for the first order system given by $u = u^{(0)} + \epsilon u^{(1)}$.

Associate $u, u^{(0)}, u^{(1)}$ to **one** the following terms (and leave the rest empty):

expression	your answer	expression	your answer
x		$x - \frac{\epsilon}{4\pi} sin(2\pi y)$	
y		$x - \frac{1}{4\pi} sin(2\pi x)$	
$-\frac{1}{4\pi} sin(2\pi x)$		$x - \frac{1}{4\pi} sin(2\pi y)$	
$-\frac{1}{4\pi} sin(2\pi y)$		$y - \frac{\epsilon}{4\pi} sin(2\pi x)$	
$-\frac{\epsilon}{4\pi} sin(2\pi x)$		$y - \frac{\epsilon}{4\pi} sin(2\pi y)$	
$-\frac{\epsilon}{4\pi} sin(2\pi y)$		$y - \frac{1}{4\pi} sin(2\pi x)$	
$x - \frac{\epsilon}{4\pi} sin(2\pi x)$		$y - \frac{1}{4\pi} sin(2\pi y)$	

Points: 4

5.11 Darcy's law

Darcy's law states

$$\mathbf{q} = -\frac{\kappa}{\mu} \left(\nabla p - \rho \mathbf{g} \right) .$$

1. What does $\mathbf{q}, \kappa, \mu, \mathbf{g}$ stand for?
2. State the SI-units of $\mathbf{q}, \kappa, \mu, \mathbf{g}$?

$\mathbf{q}$:

κ:

μ:

$\mathbf{g}$:

Points: 4

5.12 Piezometric head

An alternative form of Darcy's model in terms of the piezometric head $h(\mathbf{x})$ reads

$$\mathbf{q} = -K\nabla h \tag{9}$$

in which $K = \frac{\rho g}{\mu}\kappa$.

1. Give the missing definition for $h(\mathbf{x})$!
2. What are the unknowns in the model?
3. Which equation do we use to close the system?

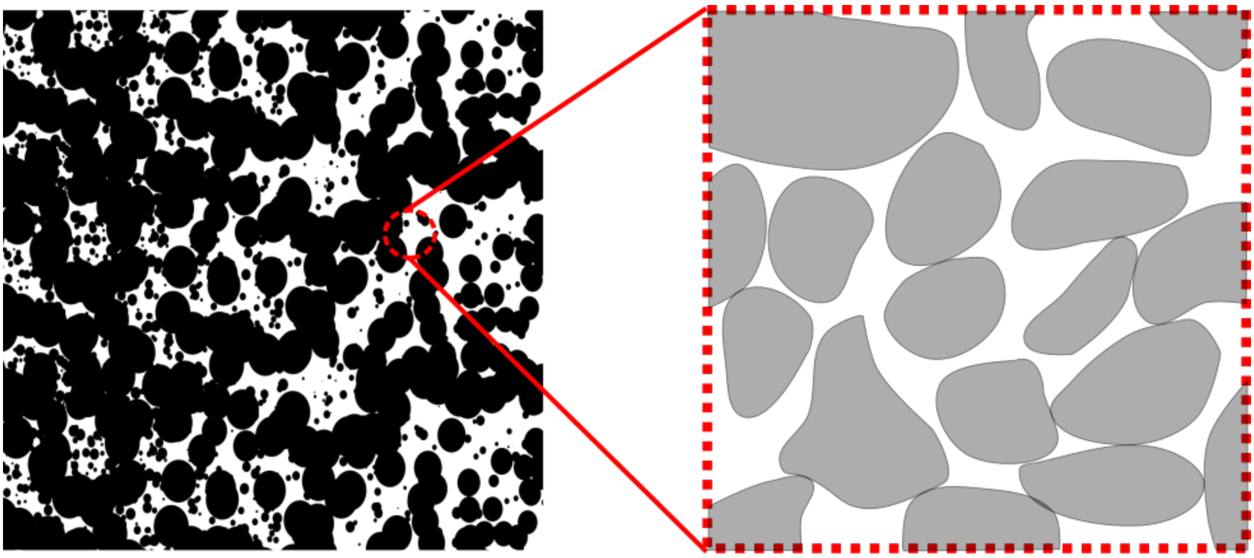
> **Tip**
>
> Continuity also holds for the Darcy velocity.

4. Combine Equation 9 and your chosen closure into a simpler model for piezometric head $h(\mathbf{x})$ only!

Points: 4

5.13 Darcy velocity

Consider a porous medium where the background medium is static.



The local velocity $\mathbf{v}$ and the Darcy velocity $\mathbf{q}$ are connected via

$$\phi \mathbf{v} = \mathbf{q}\,.$$

Assume we can measure the specific discharge through the porous medium (red circle), and use this to infer on an average macroscopic velocity through the porous medium.

1. Do we measure the local velocity $\mathbf{v}$ or the Darcy velocity $\mathbf{q}$?
2. What does ϕ stand for?
3. Consider the plot. Is $\phi > 0.5$?

☐ True
☐ False

4. The local velocity is always larger than the Darcy velocity?

☐ True
☐ False

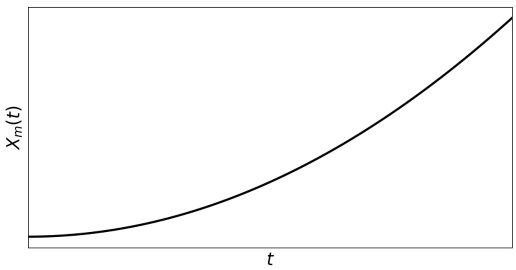
Points: 4

5.14 Stefan problem

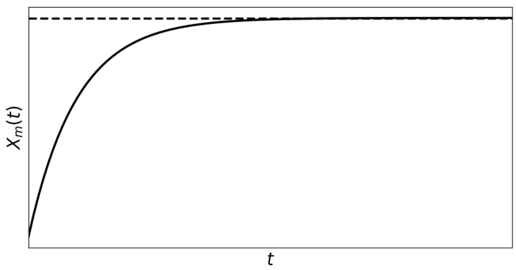
Consider a one-phase Stefan problem where we have ice water at $T_0 = 0°\mathrm{C}$ with a left boundary cooled down to $T_W = -10°\mathrm{C}$.

1. How does the interface position $X_m(t)$ propagate?

F 1



F 2



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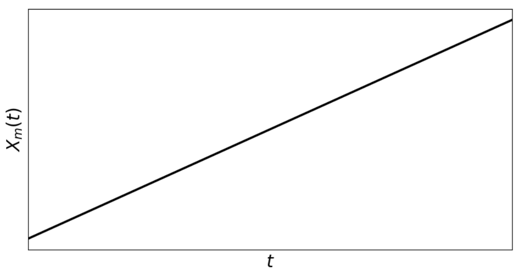
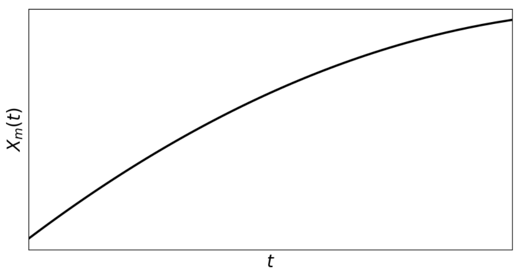


F 4



Your answer (select one):

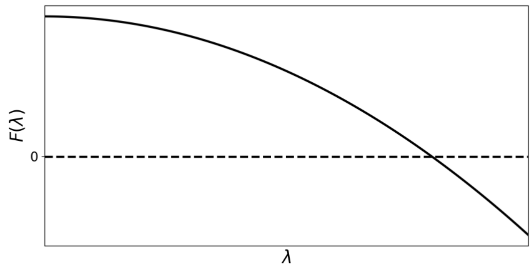
☐ F1	☐ F2	☐ F3	☐ F4

By choosing the Ansatz $X_m(t) = 2\lambda\sqrt{\alpha t}$ for the interface position, we can derive a homogeneous function

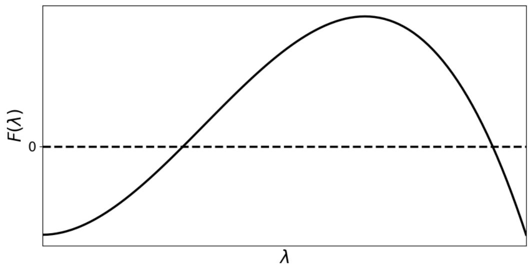
$$F(\lambda) = \mathrm{Ste}^{-1} - \lambda\sqrt{\pi}\exp\left(\lambda^2\right) erf\left(\lambda\right) = 0\,.$$

2. Which plot shows the correct trend?

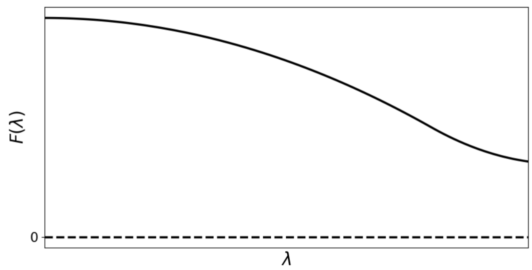
F 5



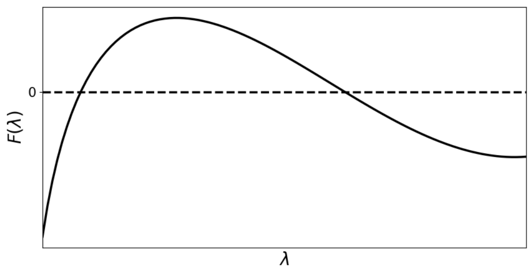
F 6



F 7



F 8



Your answer (select one):

☐ F5	☐ F6	☐ F7	☐ F8

Points: 4

5.15 Bloch equations

We focus on the exitation part of the Bloch equations. Consider a magnetic field

$$\mathbf{B}(t) = \begin{pmatrix} b_1\ \cos(\omega_0\, t) \\ -b_1\ \sin(\omega_0\, t) \\ b_0 \end{pmatrix} \quad ,$$

built from superposition of a static $\mathbf{B}_0$ and transversal magnetic field $\mathbf{B}_1(t)$. Here, b_0, b_1 are constant factors.

In the following, we assume that we know the solution of the Bloch equations for relaxation times $T_1 \to \infty$, $T_2 \to \infty$,

$$\mathbf{M}(t) = M_{z0} \begin{pmatrix} \sin(\omega_1\, t)\ \sin(\omega_0\, t) \\ \sin(\omega_1\, t)\ \cos(\omega_0\, t) \\ \cos(\omega_1\, t) \end{pmatrix} \quad ,$$

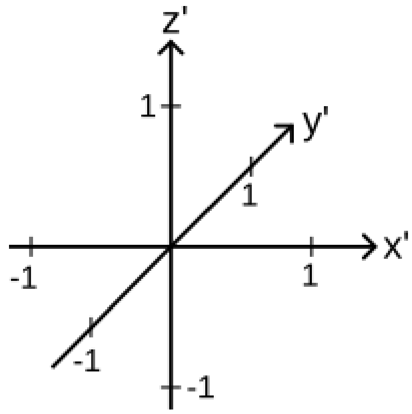
for a given initial condition $\mathbf{M}(t = 0) = (0, 0, M_{z0})^T$.

1. Write down the rotation matrix $\mathbf{R}$ (no long computation required) such that the solution $\mathbf{M}'(t) = \mathbf{R}(t)\mathbf{M}(t)$ in the rotating coordinate system reads

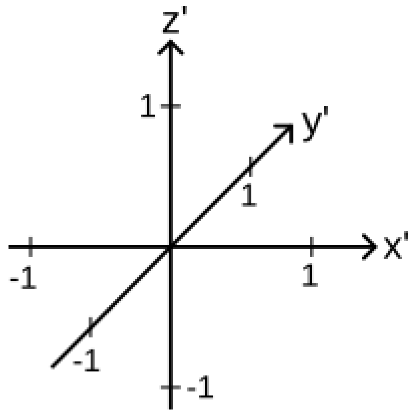
$$\mathbf{M}'(t) = M_{z0} \begin{pmatrix} \sin(\omega_1\, t) \\ 0 \\ \cos(\omega_1\, t) \end{pmatrix}$$

2. How does the magnetic field $\mathbf{B}'$ read, when written in the same rotating coordinate system?
3. **Assuming** $\omega_1 = 1$, draw $\mathbf{M}'$ and $\mathbf{B}'$ in the plots **below** by adding the unit arrows at times $t_0 = 0$ and $t_1 = \frac{1}{2}\,\pi$. Indicate which arrow belongs to which time step.

(a) $\mathbf{M}'$



(a) $\mathbf{B}'$



Points: 4