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Model-based Estimation Methods, SS 2016

Exercise 9 (June 15, 2016, 18:15 - 19:45)

Parameter Estimation

Problem 14: Parameter estimation

A model

$$y(\boldsymbol{\Theta}, u) = \Theta_1 + \Theta_2 u + \Theta_3 u^2$$

is given, where $u \in \mathbb{R}$ is the model input, $y \in \mathbb{R}$ the model prediction and $\boldsymbol{\Theta} = [\Theta_1 \quad \Theta_2 \quad \Theta_3]^T \in \mathbb{R}^3$ the vector of parameters.

To determine the parameters experiments are performed with inputs $u = (0, 0.1, 0.2, ..., 1)$. The observed measurements are
$y_n = [1.0666, 0.8935, 1.099, 1.17183, 1.039, 0.9745, 0.8231, 0.9186, 0.6855, 0.5666, 0.5699]$

The measurements are assumed to have a standard deviation
$\sigma = [0.06458, 0.1465, 0.0385, 0.1118, 0.0005, 0.0255, 0.1169, 0.0586, 0.0744, 0.0734, 0.0699]$

Calculate the weighted least squares estimate of the parameters $\boldsymbol{\Theta}$.

Use the partly-written script `Problem14`.

Problem 15: Nonlinear Parameter Estimation

Consider a system modeled by the equation

$$y = p_1 + \frac{1}{u - p_2}$$

where

$$\begin{aligned} y &\equiv \textit{measurement} \\ u &\equiv \textit{input} \\ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &\equiv \textit{parameters} \in \begin{bmatrix} 0, 10 \\ 0, 10 \end{bmatrix} \end{aligned}$$

The input(u) and output(y) from an experiment to estimate the parameters p_1 and p_2 is given in the file `parameter_estimation_data.m`.

a.) Plot the experimental data.

b.) Estimate the parameters p_1 and p_2.

The nonlinear parameter estimation problem can be set up as an optimization problem of the form

$$\min_{p_1,p_2} \Phi = \sum_i (\tilde{y}_i - y_i)^2$$

where $\tilde{y}_i$ are actual measurements and y_i are the predicted measurements calculated using the inputs u_i. Use the MATLAB function `fmincon` to solve the optimization problem.

c.) `fmincon` is a local optimization function available in MATLAB which requires an initial guess for the optimization variables - in this case p_1 and p_2. Solve the optimization problem with different initial guess values and compare the estimated values of the parameters.

Use the partly-written script `Problem15`.

Bonus Problem 6: Maximum Likelihood vs Weighted Least-Squares (pen & paper) (10 Points)

(Submit your written solutions until 18:15 on 22 June)

Consider a static model f with one-dimensional output y. The model is linear in parameters:

$$f(u, \Theta) = \sum_{j=1}^{n} x_j(u)\Theta_j$$

To estimate the unknown parameter Θ there are m measurements $\tilde{y}_i$ available:

$$\tilde{y}_i = y_i + \epsilon_i = f(u_i, \Theta) + \epsilon_i,$$

where $\epsilon_i \sim N(0, \sigma_i^2)$.

Beside least-square methods to estimate unknown parameters, there exists also a so called maximum likelihood method. The likelihood function is defined as

$$L(\bar{\Theta}) = p(\tilde{y}|\bar{\Theta}, u) = \prod_{i=1}^{m} p(\tilde{y}_i|\bar{\Theta}, u_i),$$

where $p(\tilde{y}_i|\bar{\Theta}, u_i)$ is the probability density function to get the measurement $\tilde{y}_i$ if we use u_i as inputs and $\bar{\Theta}$ as parameters in the model. For $\epsilon_i \sim N(0, \sigma_i^2)$ the probability density function

$$p(\tilde{y}_i|\bar{\Theta}, u_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(\tilde{y}_i - \bar{y}_i)^2}{2\sigma_i^2}},$$

where $\bar{y}_i = f(u_i, \bar{\Theta})$. The optimal parameters in the sense of the maximum likelihood methods are the parameters $\bar{\Theta}$ for which the probability to get $\tilde{y}$ as measurements is maximal. i.e

$$\hat{\Theta}_{ML} = \arg\max_{\bar{\Theta}} L(\bar{\Theta}).$$

Show that for models which are linear in parameters the maximum likelihood solution $\hat{\Theta}_{ML}$ of independent and normally distributed outputs is equivalent to a weighted least-squares solution with corresponding weight matrix. Give the weight matrix of the weighted least-square method.