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Model-based Estimation Methods, SS 2016

Exercise 11 (July 6, 2016, 18:15 - 19:45)

State Estimation

Problem 16: State Estimation of a Two-Tank Process Using Kalman Filter (Matlab)

Consider the two-tank process in Fig. 1. The mass balances for tank 1 and tank 2 are given by the equations

$$\begin{aligned} A_1 \frac{dh_1}{dt}(t) &= F_{in}(t) - F_1(t) \\ A_2 \frac{dh_2}{dt}(t) &= F_1(t) - F_2(t), \end{aligned}$$

where A_1 and A_2 are cross sectional areas of tank 1 and tank 2, respectively. R_1 and R_2 describe the resistance to the flow. We assume a linear resistance to flow, i.e.

$$\begin{aligned} F_1(t) &= h_1(t)/R_1 \\ F_2(t) &= h_2(t)/R_2. \end{aligned}$$

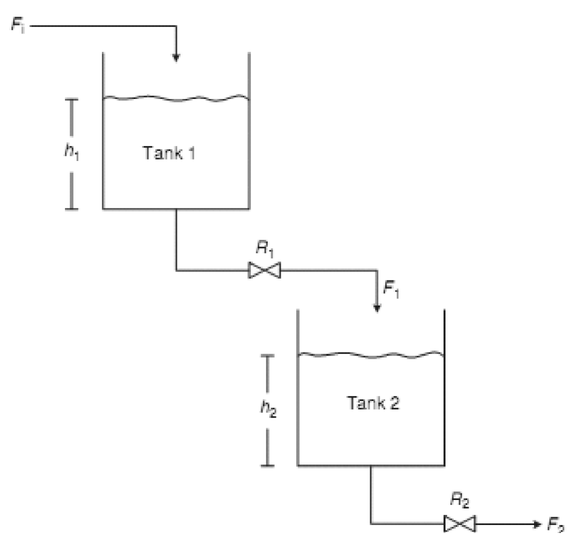

Figure 1: Tank cascade

a.) List the states, inputs, parameters and measurements of the two-tank system.

b.) Rewrite the process equations in the general LTI form with Gaussian noises $w_x \in \mathbb{R}^2$ and $v_x \in \mathbb{R}$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w_x(t), \; x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) + v_y(t). \end{aligned}$$

Assume that only h_2 can be measured and F_{in} is the input flow. Consider:

$$\begin{aligned} R_1 &= R_2 = 1, \text{ resistances to the flows} \\ A_1 &= 2\;[m^2], \text{ cross sectional area of tank 1} \\ A_2 &= 5\;[m^2], \text{ cross sectional area of tank 2} \\ w_x &\sim N(0, \sigma_x^2), \sigma_x^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{process noise} \\ v_x &\sim N(0, \sigma_y^2), \sigma_y^2 = 0.01, \text{ measurement noise} \\ h_1(0) &= 0\;[m], \text{ initial height in tank 1} \\ h_2(0) &= 1\;[m], \text{ initial height in tank 2} \\ F_{in}(t) &= \sin(0.1t) + 1\;[m^3/s], \text{ input signal} \\ T &= 0.001\;[s], \text{ sample time} \\ t_{end} &= 40\;[s], \text{ simulation time} \end{aligned}$$

and simulate the cascade (Hint: use the mfile *CascadeSimulator.m*).

c.) Discretize the continuous LTI using the Euler method and check the observability of the discrete LTI system (Hint: use the Kalman's criterion for observability).

d.) Implement the discrete Kalman filter, run it after the simulation with `CascadeSimulator.m`. Plot the results for simulated h_1, h_2 and filtered $h_{1,\mathrm{KF}}$, $h_{2,\mathrm{KF}}$ in one plot. Use the following filter parameters for the Kalman filter implementation:

$$\begin{aligned} \hat{x}^+[0] &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}' \\ P^+[0] &= \mathrm{diag}(0.01,\, 0.01) \\ Q &= \mathrm{diag}(0.1,\, 0.1) \\ R &= 1 \end{aligned}$$

For your solution, edit the file `Kalman_filter_two_tank` and use the file `CascadeSimulator.m`.

Bonus Problem 10: Luenberger observer (Points(2)) - Pen & Paper

(Submit by July 13 18:15)

The equations of the discrete time Luenberger observer can be written as:

$$\begin{aligned}\hat{x}[i+1] &= A_d\hat{x}[i] + L_d(y[i] - \hat{y}[i]) + B_d u[i] \\ \hat{y}[i] &= C_d\hat{x}[i] + D_d u[i]\end{aligned}$$

Where L_d is the observer gain.

Derive an expression for observer error $e[i+1]$ as a function of $e[i]$, where $e[i] = x[i] - \hat{x}[i]$. What condition must be satisfied so that the Luenberger Observer serves the purpose of a state estimator?

Problem 16: Luenberger observer for the two-tank system. (Matlab)

Implement a discrete-time Luenberger observer for the two tank system. Plot the estimated and true values of the heights in the two tanks. Assume the estimated initial states to be $\hat{x}[0] = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$. (*Hint:* Use the MATLAB function `place` to determine the gain of the observer.)

For your solution, edit the file `LuenbergerObserver_2tank` and use the file `CascadeSimulator.m`.

Bonus Problem 11: Kalman Gain Derivation (Points 8)- Pen & Paper

(Submit by July 13 18:15)

Consider a discrete time LTI system of the form

$$\begin{aligned}x[k+1] &= A_d x[k] + B_d u[k] + w[k] \\ \tilde{y}[k] &= C_d x[k] + D_d u[k] + v[k]\end{aligned}$$

Where

$$
\begin{aligned}
x &\equiv \text{States} \\
u &\equiv \text{Inputs} \\
\tilde{y} &\equiv \text{Outputs} \\
w &\equiv \text{State noise} \sim \mathit{N(0,Q)} \\
v &\equiv \text{Measurement noise} \sim \mathit{N(0,R)}
\end{aligned}
$$

It is assumed that the state noise (w) and the measurement noise(v) are neither auto-correlated nor cross-correlated.

The measurement update step of the discrete time Kalman Filter is

$$\hat{x}^+[k] \;=\; \hat{x}^-[k] + K[k](\tilde{y} - C_d\hat{x}^-[k] - D_d u[k])$$

Where

$$
\begin{aligned}
\hat{x}^- &\equiv \text{Apriori state estimate} \\
\hat{x}^+ &\equiv \text{Aposteriori state estimate} \\
K[k] &\equiv \text{Kalman Gain matrix}
\end{aligned}
$$

Derive expressions for

1. $P^+[k]$, where $P^+[k] = E[(x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])']$, in terms of $K[k]$, C_d and R.

2. Kalman Gain matrix ($K[k]$) which minimizes $trace(P^+[k])$, in terms of $P^-[k]$, C_d, R.

3. $P^-[k+1]$ in terms of $P^+[k]$, A_d and Q.