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Model-based Estimation Methods, SS 2016

Solution - Tutorial 10 (July 06, 2016)

State Estimation

Solution of Problem 16

a.) The states of the system are h_1 and h_2.

The inputs of the system is F_{in}.

The parameters of the system are R_1, R_2, A_1 and A_2.

The output of the system is h_2.

b.) In LTI form the model equations of the two tank system can be rewritten as

$$
\begin{aligned}
\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} &= \begin{bmatrix} \frac{-1}{R_1 \cdot A_1} & 0 \\ \frac{1}{R_1 \cdot A_1} & \frac{-1}{R_2 \cdot A_2} \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot F_{in} \\
h_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}
\end{aligned}
$$

Solution of Bonus Problem 10

The error at instant $i+1$ can be written as

$$
\begin{aligned}
e[i+1] &= x[i+1] - \hat{x}[i+1] \\
&= A_d x[i] + B_d u[i] - (A_d \hat{x}[i] + B_d u[i] + L_d(C_d x[i] - C_d \hat{x}[i])) \\
&= (A_d - L_d C_d)(x[i] - \hat{x}[i]) \\
&= (A_d - L_d C_d) e[i]
\end{aligned}
$$

In order for the error to decay with time the eigenvalues of the matrix $(A_d - L_d C_d)$ must lie within the unit circle.

Solution of Bonus Problem 11

1. $P^+[k]$ is defined as

$$
P^+[k] = E[(x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])']
$$

Substituting $\hat{x}^+[k] = \hat{x}^-[k] + K[k](\tilde{y} - C_d\hat{x}^-[k] - D_d u[k])$

$$\begin{aligned} P^+[k] &= E[(x[k] - (\hat{x}^-[k] + K[k](\tilde{y} - C_d\hat{x}^-[k] - D_d u[k]))) \\ &\quad (x[k] - (\hat{x}^-[k] + K[k](\tilde{y} - C_d\hat{x}^-[k] - D_d u[k])))'] \end{aligned}$$

Since $\tilde{y}[k] = C_d x[k] + D_d u[k] + v[k]$

$$\begin{aligned} P^+[k] &= E[(x[k] - (\hat{x}^-[k] + K[k](C_d x[k] - C_d\hat{x}^-[k] + v[k]))) \\ &\quad (x[k] - (\hat{x}^-[k] + K[k](C_d x[k] - C_d\hat{x}^-[k] + v[k])))'] \\ &= E[((I - K[k]C_d)(x[k] - \hat{x}^-[k]) - K[k]v[k]) \\ &\quad ((I - K[k]C_d)(x[k] - \hat{x}^-[k]) - K[k]v[k])'] \\ &= E[(I - K[k]C_d)(x[k] - \hat{x}^-[k])(x[k] - \hat{x}^-[k])'(I - K[k]C_d)'] \\ &\quad -E[(I - K[k]C_d)(x[k] - \hat{x}^-[k])v'[k]K'[k]] - E[K[k]v[k](x[k] - \hat{x}^-[k])'(I - K[k]C_d)'] \\ &\quad +E[K[k]v[k]v'[k]K'[k]] \end{aligned}$$

Since $K[k]$ and C_d are constants and using the property

$$E[kx] = kE[x] \tag{1}$$

Where

$$\begin{aligned} x &\equiv \text{Stochastic variable} \\ k &\equiv \text{Constant} \end{aligned}$$

We now have,

$$\begin{aligned} P^+[k] &= (I - K[k]C_d)E[(x[k] - \hat{x}^-[k])(x[k] - \hat{x}^-[k])'](I - K[k]C_d)' \\ &\quad -(I - K[k]C_d)E[(x[k] - \hat{x}^-[k])v'[k]]K'[k] - K[k]E[v[k](x[k] - \hat{x}^-[k])'](I - K[k]C_d)' \\ &\quad +K[k]E[v[k]v'[k]]K'[k] \end{aligned}$$

We know,

$$\begin{aligned} E[(x[k] - \hat{x}^-[k])(x[k] - \hat{x}^-[k])'] &= P^-[k] \\ E[(x[k] - \hat{x}^-[k])v'[k]] &= 0 \qquad \text{(Since } x[k] \text{ and } v[k] \text{ are uncorrelated)} \\ E[v[k]v'[k]] &= R \end{aligned}$$

Therefore,

$$P^+[k] = (I - K[k]C_d)P^-[k](I - K[k]C_d)' + KRK'$$

2. According to the first order necessary condition for optimality, we have

$$\frac{d(trace(P^+[k]))}{dK[k]} = 0$$

Therefore,

$$0 = trace((I - K[k]C_d)P^-[k](-C_d)' + (-C_d)P^-[k](I - K[k]C_d)' + KR + RK')$$

Since the trace of a matrix product **AB** is independent of the order of **A** and **B**, we can write the equation above as

$$\begin{aligned} 0 &= trace(2(I - K[k]C_d)P^-[k](-C_d)' + 2KR) \\ 0 &= trace(-P^-[k]C_d' + K[k]C_dP^-[k]C_d' + KR) \\ 0 &= trace(-P^-[k]C_d' + K[k](C_dP^-[k]C_d' + R)) \end{aligned}$$

Therefore, the Kalman gain matrix $K[k]$ which minimizes the $trace(P^+[k])$ is given by

$$K[k] = P^-[k]C_d'(C_dP^-[k]C_d' + R)^{-1}$$

3.

$$P^-[k+1] = E[(x[k+1] - \hat{x}^-[k+1])(x[k+1] - \hat{x}^-[k+1])']$$

Substituting $x[k+1] = A_dx[k] + B_du[k] + w[k]$ and $\hat{x}^-[k+1] = A_d\hat{x}^+[k] + B_du[k]$, we now have

$$\begin{aligned} P^-[k+1] &= E[(A_dx[k] + w[k] - A_d\hat{x}^+[k])(A_dx[k] + w[k] - A_d\hat{x}^+[k])'] \\ &= E[(A_d(x[k] - \hat{x}^+[k]) + w[k])(A_d(x[k] - \hat{x}^+[k]) + w[k])'] \end{aligned}$$

Making use of the property in Eq.1, we have

$$\begin{aligned} P^-[k+1] &= A_dE[(x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])']A_d' \\ &\quad + A_dE[(x[k] - \hat{x}^+[k])w'[k]] + E[w[k](x[k] - \hat{x}^+[k])']A_d' \\ &\quad + E[w[k]w'[k]] \end{aligned}$$

We know,

$$\begin{aligned} E[(x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])'] &= P^+[k] \\ E[(x[k] - \hat{x}^-[k])w'[k]] &= 0 \qquad \text{(Since } x[k] \text{ and } w[k] \text{ are uncorrelated)} \\ E[w[k]w'[k]] &= Q \end{aligned}$$

Therefore,

$$P^-[k+1] = A_dP^+[k]A_d' + Q$$