13. JUL. 2016 <Model Basted Estimation method

Bonus problem 10 & 11 >

· JAEYONG JUNG 359804

· SANGWOOK LEE 362209

· BP 10. Luenberger observer

$$
\begin{aligned}
e[i+1] &= x[i+1] - \hat{x}[i+1] \\
&= Ax[i] + \cancel{Bu[i]} - \{ A\hat{x}[i] + L(y[i] - \hat{y}[i]) + \cancel{Bu[i]} \} \\
&= Ax[i] - A\hat{x}[i] - L(Cx[i] + \cancel{Du[i]} - (C\hat{x}[i] + \cancel{Du[i]})) \\
&= (A - LC)(x[i] - \hat{x}[i]) \\
&= \underline{(A - LC)\, e[i]}
\end{aligned}
$$

All eigen values of $(A-LC)$ must have negative real parts to serve the purpose of a state estimator.

i.e. $e(t) \to 0$ for $t \to \infty$

$\to$ $\hat{x}(t) \to x(t)$ for $t \to \infty$

BP. 11. Kalman Gain Derivation

1. $P^+[k]$, where $P^+[k] = E[(x[k] - \hat{x}[k])(x[k] - \hat{x}^+[k])')$

$$\hat{x}^+[k] = \hat{x}^-[k] + K[k](\hat{y}[k] - C_d \hat{x}^-[k] - D_d u[k])$$
$$= \hat{x}^-[k] + K[k](C_d x[k] + \cancel{D_d u[k]} + v[k] - C_d \hat{x}^-[k] - \cancel{D_d u[k]})$$
$$= \hat{x}^-[k] + K[k](C_d(x[k] - \hat{x}^-[k]) + v[k])$$

$$P^+[k] := E((x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])^T), \quad - ①$$

so we need $(x[k] - \hat{x}^+[k])$.

$$x[k] - x^+[k] = x[k] - \hat{x}^-[k] - K[k](C_d(x[k] - \hat{x}^-[k]) + v[k])$$
$$= I \cdot (x[k] - \hat{x}^-[k]) - K[k](C_d(x[k] - \hat{x}^-[k])) - K[k]v[k]$$
$$= (I - K[k] \cdot C_d)(x[k] - \hat{x}^-[k]) - K[k]v[k] \quad - ②$$

now we plug in ② to ①,

$$P^+[k] = E\{(I - K[k]C_d)(x[k] - \hat{x}^-[k])(I - K[k]C_d)(x[k] - \hat{x}^-[k]))^T + K[k]v[k](K[k]v[k])^T\}$$
$$= E\{(I - K[k]C_d)(x[k] - \hat{x}^-[k])(x[k] - \hat{x}^-[k])^T(I - K[k]C_d)^T\}$$
$$= (I - K[k]C_d)\, E\{\underbrace{(x[k] - \hat{x}^-[k])(x[k] - \hat{x}^-[k])^T}_{P^-[k]}\}(I - K[k]C_d)^T + K[k]\underbrace{v[k]v[k]^T}_{Q_k}K[k]^T$$
$$+ K[k] \cdot Q_k K[k]^T$$

$$= \underline{(I - K[k]C_d)P^-[k](I - K[k]C_d)^T + K[k]Q_k K[k]^T}$$

2. Kalman Gain matrix $(K[k])$ which minimizes trace $(P^+[k])$

$J = \text{trace}\,(P^+[k])$, Necessary optimality onditions for

$$J \Leftrightarrow \frac{\partial J}{\partial K} = 0$$

$$\frac{\partial}{\partial K} P^+[k] = \frac{\partial}{\partial K}\left[(I - K[k]\, C_d)\, P^-[k]\, (I - C_d^T K[k]^T) + K[k]\, Q_k\, K[k]^T\right]$$

$$= (I - K[k]\, C_d)\, P^-[k]\, (-C_d^T) - (I - K[k]\, C_d)\, P^-[k]^T C_d^T$$

$$+ K[k]\, Q_k + K[k]\, Q_k^T$$

$$= 2\{(I - K[k]\, C_d)\, P^-[k]\, (-C_d^T) + \underline{K[k]\, Q_k}\} = 0$$

$$\Leftrightarrow K[k]\,(-C_d \cdot P^-[k]\,(-C_d^T) + Q_k) - P^-[k]\, C_d^T = 0$$

$$\Leftrightarrow \underline{K[k] = P^-[k]\, C_d^T (C_d P^-[k]\, C_d^T + R_k)^{-1}}$$

3. $P^-[k+1] = E\{(x[k+1] - \hat{x}^-[k+1])(x[k+1] - \hat{x}^-[k+1])^T\}$

$$x[k+1] = A_d \cdot x[k] + B_d \cdot u[k] + w[k]$$
$$\hat{x}^-[k+1] = A_d \cdot \hat{x}^+[k] + B_d \cdot u[k]$$

$$x[k+1] - \hat{x}^-[k+1] = A_d\,(x[k] - \hat{x}^+[k] + w[k])$$

$$P^-[k+1] = E\{A_d (x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])^T A_d^T$$
$$\underline{+ A_d (x[k] - \hat{x}^+[k])\, w[k]^T + w[k]\,(x[k] - \hat{x}^+[k])^T A_d^T}$$
$$+ w[k]\, w^T[k]\}$$

$(x[k] - \hat{x}^+[k])$ is uncorrelated with $w[k]$ $\rightarrow 0$

$$= A_d\, E\{(x[k] - \hat{x}^+[k])(x[k] - \hat{x}^+[k])^T\}\, A_d^T + E\{w[k]\, w^T[k]\}$$

$$= \underline{A_d\, P^+[k]\, A_d^T + R[k]}$$