**Lehrstuhl für Prozesstechnik**
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**Lehrstuhl für Numerische Mathematik**
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# Model-based estimation methods, summer semester 2016

**Sheet 8** (June 8th, Wednesday 18:15 - 19:45)

## Problem 13 : Discrepancy principle, L-curve

We reconsider the least squares problem from Sheet 6, Problem 11.

$$g(t) = \xi_1 t + \xi_2 e^t + \xi_3 t^3 + \xi_4 \sin t, \qquad \xi^* = (\xi_1, \xi_2, \xi_3, \xi_4)^T := (1.2,\ 0.6,\ 1.6,\ 0.9)^T.$$

Using discrete points $t_i = i\Delta t$, $i = 1, \ldots, 6$ for $\Delta t = 0.15$, we get an (exact) data vector $y \in \mathbb{R}^6$ with $y_i = g(t_i)$. We choose the perturbation $\delta y = 10^{-2}\,(-1, 1, 1, -0.5, -2, 1)^T$, yielding perturbed data $\tilde{y} = y + \delta y$ and consider the corresponding least-squares problem

$$\|A\tilde{\xi} - \tilde{y}\|_2 \to \min\,.$$

For the solution of this ill-posed problem we consider Tikhonov regularization, choosing the range of the regularization parameter $\alpha \in [10^{-4}, 10^1]$.

a.) Plot the discrepancy $\|A\tilde{\xi}_\alpha - \tilde{y}\|_2$ as a function of $\alpha$. What is a good choice of $\alpha$ according to the discrepancy principle? For your chosen $\alpha$, plot the parameters $\xi^*, \tilde{\xi}_\alpha$ and the data $y, \tilde{y}, \bar{y} = A\tilde{\xi}_\alpha$ in respective figures. Discuss your results.

b.) If the magnitude of the data error is not known, the discrepancy principle cannot be applied. Then heuristic approaches like the L-curve may be the method of choice which do not require any a-priori knowledge on the data error.

To this end, we plot the values $(\|A\tilde{\xi}_\alpha - \tilde{y}\|_2, \|\tilde{\xi}_\alpha\|_2)$ in a `loglog` plot. Plotting also the values of $\alpha$ using `text(`$x, y$`,num2str(`$\alpha$`))` to find a reasonable choice of $\alpha$.

## Bonus Problem 5 : Iterative method

(Submit your program till 18:15 on 15th June in the L2P learning room.)

We reconsider the backward heat conduction problem from Sheet 6, Problem 10, with $n = 100$ unknowns Using the usual notation, we have to solve the linear problem

$$A\tilde{\xi} = \tilde{y} \qquad \text{with} \quad A = \exp(-TC), \quad \tilde{\xi} = \vec{u}_0^{\,\text{est}}, \quad \tilde{y} = \vec{u}_T^{\,\text{meas}}$$

where the measured final temperature $\vec{u}_T^{\text{meas}}$ is generated by $u(T)$ adding a perturbation `1e-2*randn(n,1)` with normal distribution and variance $\sigma = 10^{-2}$. Note that the measurement error has variance $\sigma = 10^{-2}$, leading to $\|\tilde{y} - y\|_2 \approx \sigma\sqrt{n} = 10^{-1}$. To solve the problem iteratively and for the purpose of regularization, we apply 250 iterations of the Landweber method

$$\tilde{\xi}_{k+1} = \tilde{\xi}_k - \beta A^T(A\tilde{\xi}_k - \tilde{y}), \qquad k = 0, 1, 2, \ldots$$

with $\tilde{\xi}_0 = 0$ and $\beta = 1$. By choosing a suitable stopping index $k^*$ (which plays the role of a regularization parameter) we will obtain a regularized solution $\tilde{\xi}_{k^*}$.

a.) Use the discrepancy principle to obtain a good stopping index $k^*$. Plot the exact and estimated initial temperature $\xi^*, \tilde{\xi}_{k^*}$ as well as the final temperature data $y, \tilde{y}, \bar{y} = A\tilde{\xi}_{k^*}$ in respective figures. Discuss your results.

b.) Now we want to choose a stopping index $k^*$ by means of the L-curve. To this end, plot the values $(\|A\tilde{\xi}_k - \tilde{y}\|_2, \|\tilde{\xi}_k\|_2)$ for $k = 1, \ldots, 250$ in a `loglog` plot. What do you observe?

c.) Plot another L-curve, replacing $\|\tilde{\xi}_k\|_2$ by $\|C\tilde{\xi}_k\|_2$. Note that the $C$-norm will penalize oscillations in the estimate $\tilde{\xi}$ (remember: $C$ is a discrete Laplacian operator). For the obtained index $k^*$, plot initial and final temperatures in respective plots as in a.), and discuss the results.