Lehrstuhl für Prozesstechnik
Dr. Ing. Adel Mhamdi

Lehrstuhl für Numerische Mathematik
Prof. Dr. A. Reusken

Model-based estimation methods, summer semester 2016

Sheet 7 (30th May, Monday 10:15 - 11:45)

Problem 12 : SVD, TSVD, Tikhonov

The matrix A is given by

$$A = \begin{pmatrix} 25.6 & 30.8 \\ 23.2 & 32.6 \\ 10.4 & 17.2 \end{pmatrix}$$

with

$$A^T A = \begin{pmatrix} 1301.76 & 1723.68 \\ 1723.68 & 2307.24 \end{pmatrix}.$$

The SVD of the matrix A is $A = U\Sigma V^T$, where

$$U = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}.$$

a.) Show that the singular values of A are $\sigma_1 = 60$ and $\sigma_2 = 3$;

b.) What is the condition number $\kappa_2(A)$? Can you also show the rank of A?

Let

$$y = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}.$$

Consider the linear least-squares problem $\min_{x \in \mathbb{R}^2} \|Ax - y\|_2$.

c.) Does this least-squares problem have a unique solution? Explain your answer.

d.) Compute the solution(s) of this least-squares problem. (Hint: Use the SVD!)

e.) Consider the TSVD-regularization method, how should you choose the parameter α if you want to filter the smaller singular value? Apply the α you choose on the current problem and compute the corresponding regularized solution.

f.) Compute the regularized solution x_α by means of the Tikhonov regularization with $\alpha = 5$.

Bonus Problem 4 : General inverse

(Submit till 18:15 on 8th June in the tutorial class) (3+4+3=10 Points)

Consider the data y and perturbed data $\tilde{y} = y + \delta y$. Prove the following assertions:

a.) The solution $\xi = A^\dagger y$ of the least squares problem

$$\|A\xi - y\|_2 = \min_{v \in \mathbb{R}^n} \|Av - y\|_2 \quad \text{with } \|\xi\|_2 \text{ minimal}$$

has the representation $\xi = \sum_{i=1}^r \sigma_i^{-1}(u_i^T y)v_i$.

b.) For $\tilde{\xi} = A^\dagger \tilde{y}$ the following error bound holds:

$$\|\tilde{\xi} - \xi\|_2 \leq \sigma_r^{-1} \|\tilde{y} - y\|_2.$$

Moreover, the inequality reaches equal if the perturbation δy is in direction of u_r.

c.) Given the least-square problem

$$\min_{x \in \mathbb{R}^3} \left\| \frac{1}{27} \begin{pmatrix} 16 & 52 & 80 \\ 44 & 80 & -32 \\ -9 & -36 & -72 \\ -16 & -16 & 64 \end{pmatrix} x - \begin{pmatrix} 4 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\|_2,$$

$$U = \frac{1}{45} \begin{pmatrix} 32 & 5 & 24 & -20 \\ 4 & 40 & 3 & 20 \\ -27 & 0 & 36 & 0 \\ 16 & -20 & 12 & 35 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V^T = \frac{1}{9} \begin{pmatrix} 1 & 4 & 8 \\ 4 & 7 & -4 \\ -8 & 4 & -1 \end{pmatrix}$$

are corresponding matrices for SVD, please compute the best-approximate-solution x.