M.Sc. Simulation Sciences, Summer Semester 2016

# Fast Iterative Solvers

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Project 2

Due: August 2, 2016, by midnight

## Summary


We implement a multigrid solver for the Poisson equation

$$\begin{aligned} -\nabla^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\Omega = (0,1) \times (0,1)$, using a Finite Difference discretization on a Cartesian Grid,

$$\mathcal{G}_h := \{(ih, jh) : i, j = 0, \ldots, N;\ hN = 1\}.$$

This means, find $u_{i,j} \approx u(x_i, y_j)$, such that

$$\begin{aligned} -f_{i,j} &= \frac{1}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + \frac{1}{h^2}\left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right), && i, j = 1, \ldots, N-1 \\ u_{i,j} &= 0, && \text{otherwise.} \end{aligned}$$

Use $f(x, y) = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$. For this choice, the solution is $u(x, y) = \sin(2\pi x)\sin(2\pi y)$.


## Instructions

- Use meshes with $N = 2^n$ for the fine mesh. For the next coarser mesh use $N^c := N/2$. This means that points in the coarse mesh will also be points in the fine mesh, while every other point in the fine mesh is deleted.
- Mandatory: Implement the Gauss-Seidel Smoother, restriction, and prolongation as defined in the tutorials
- Optional: you may implement and test other choices for these operators
- Use W-cycles (i.e. $\gamma = 2$, as discussed in class).
- You should use as many multigrid levels as possible. Use the same iterative solver on each mesh level. (Recall: One should solve exactly on the coarsest mesh. If there is only one interior grid point on the coarsest mesh, GS becomes exact in one step!)
- Plot the convergence using the measure $||\mathbf{r}^{(m)}||_\infty / ||\mathbf{r}^{(0)}||_\infty$ against multigrid iterations $m$ for meshes with $n = 4$, $n = 7$ (resulting in $N = 16$, and $N = 128$). Use a semi-log scale, and do this for

1. $\nu_1 = \nu_2 = 1$
2. $\nu_1 = 2,\ \nu_2 = 1$

where the $\nu_i$ are the Gauss-Seidel pre- and post smoothing operations, as discussed in class. $\mathbf{r}^{(m)}$ is the residual evaluated at the $m^{th}$ iteration, and $\mathbf{r}^{(0)}$ is the residual evaluated with the initial guess (you may use $\mathbf{u} \equiv 0$ as initial guess). The norm $||\cdot||_\infty$ is defined in the usual way, i.e. $||\mathbf{r}||_\infty = \max_{i,j} |r_{i,j}|$, where $(i,j)$ ranges over all interior points.

- you may optionally want to do more numerical experiments. For instance, you may want to verify the claim that it doesn't make sense to do too many smoothing iterations, by measuring the convergence against *run-time* (instead of iteration), and increase the number of smoothing steps $\nu$.