



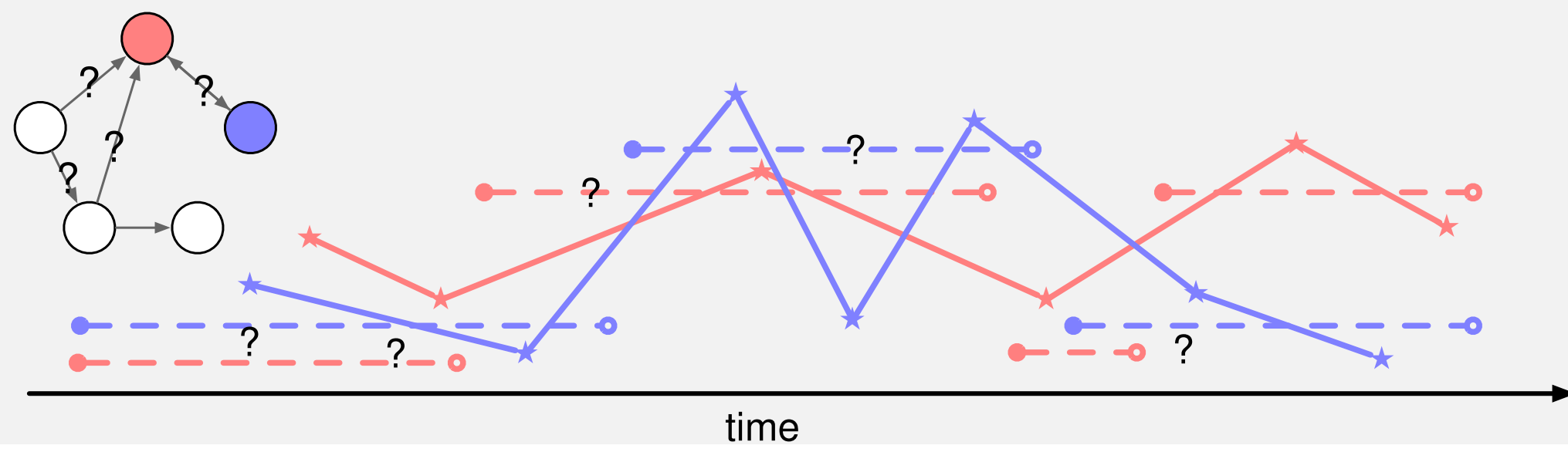
Scalable Structure Learning of Continuous-Time Bayesian Networks from Incomplete Data



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Motivation

- Scalable learning of directed networks from time series data
- If data is noisy there exist two bottlenecks:
 - Latent state estimation (**exponential**)
 - Structure search (**super-exponential**)



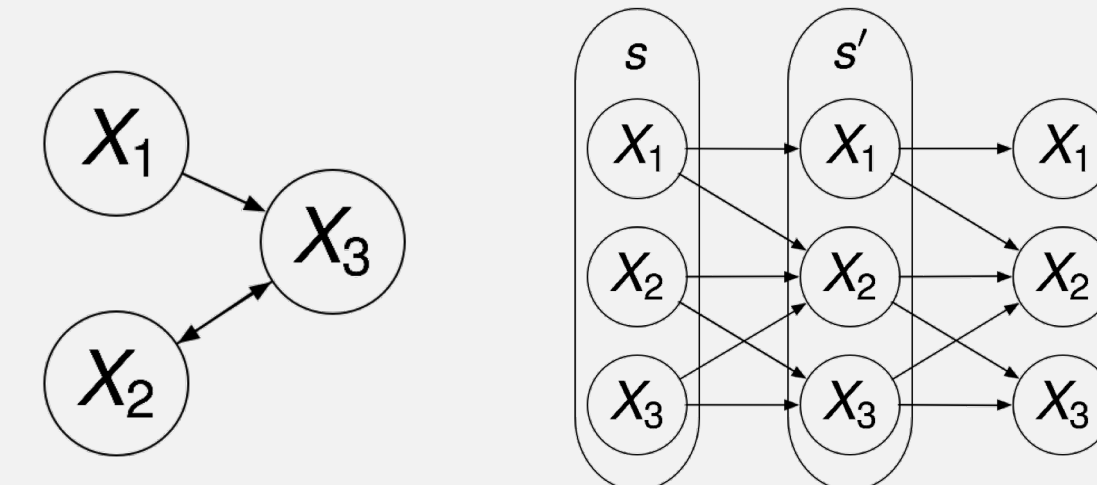
Background: Continuous-time Bayesian Nets (CTBN)

- Factorized Markov Chain
 - Can be expressed in terms of individual processes:

$$\{X_i(t)\}_{i:1,\dots,N} \quad U_i(t) \equiv \{X_j(t)\}_{j \in \text{par}(i)}$$
 - Dynamics in continuous-time via **conditional intensities**:

$$r_i(x', x | u) = \lim_{h \rightarrow 0} \frac{P_i(X_i(t+h) = x' | X_i(t) = x, U_i(t) = u)}{h}$$
- Path likelihood, $P(X_{[0,T]} | \{r_i\}_{i:1,\dots,N}) = \prod_{i=1}^N \prod_{x,x',u} r_i(x, x' | u)^{M_i(x,x' | u)} e^{-T_i(x | u)}$
 in terms of **sufficient statistics**:

$$M_i(x, x' | u) \quad T_i(x | u)$$
 = number of transitions = time spend in configuration



Background: CTBN structure learning

- Realization of conditional intensities unimportant for structure learning
- Marginalization possible analytically under Gamma-prior

$$r_i(x, x' | u) \sim \text{Gam}(\alpha, \beta)$$

$$P(X_{[0,T]} | \mathcal{G}) \propto \prod_i \prod_{x,x',u} \Gamma(\bar{\alpha}_i(x, x' | u)) \bar{\beta}_i(x | u)^{-\bar{\alpha}_i(x, x' | u)}$$

$$\bar{\alpha}_i(x, x' | u) = M_i(x, x' | u) + \alpha, \quad \bar{\beta}_i(x | u) = T_i(x | u) + \beta$$

Model: Mixture Conditional Intensities

- Central assumption:** Conditional intensities can be written as a mixture of conditional intensities of different parent-support

$$r_i(x, x' | u) = \sum_{m \in \mathcal{P}(\text{par}_{\mathcal{G}}(i))} \pi_i(m) r_i(x, x' | u_m)$$

- Probability for an edge can be represented via these mixtures

$$p(e_{ij} = 1) = \sum_{m \in \mathcal{P}(\text{par}_{\mathcal{G}}(j))} \pi_j(m) 1(i \in m)$$

- A **marginal likelihood lower-bound** can be computed

$$P(X_{[0,T]} | \pi) \geq \prod_i \prod_{m, u_m, x, x'} \frac{\Gamma(\bar{\alpha}_i(x, x' | u_m))}{\bar{\beta}_i(x | u_m)^{\bar{\alpha}_i(x, x' | u_m)}}$$

$$\bar{\alpha}_i(x, x' | u_m) = \pi_i(m) M_i(x, x' | u_m) + \alpha,$$

$$\bar{\beta}_i(x | u_m) \equiv \pi_i(m) T_i(x | u_m) + \beta$$

Method

- Based on marginal likelihood we define our **objective function** for structure learning (log-posterior lower-bound)

$$\mathcal{F}_i[\pi] \equiv \sum_{m, u_m, x, x'} \{ \ln \Gamma(\bar{\alpha}_i(x, x' | u_m)) - \bar{\alpha}_i(x, x' | u_m) \ln \bar{\beta}_i(x | u_m) \} + \ln \text{Dir}(\pi_i | c_i)$$
- Optimal mixture (structure) can be computed via: $\pi_i^* = \arg \max_{\pi_i \in \Delta_i} \{\mathcal{F}_i[\pi]\}$

Complete data

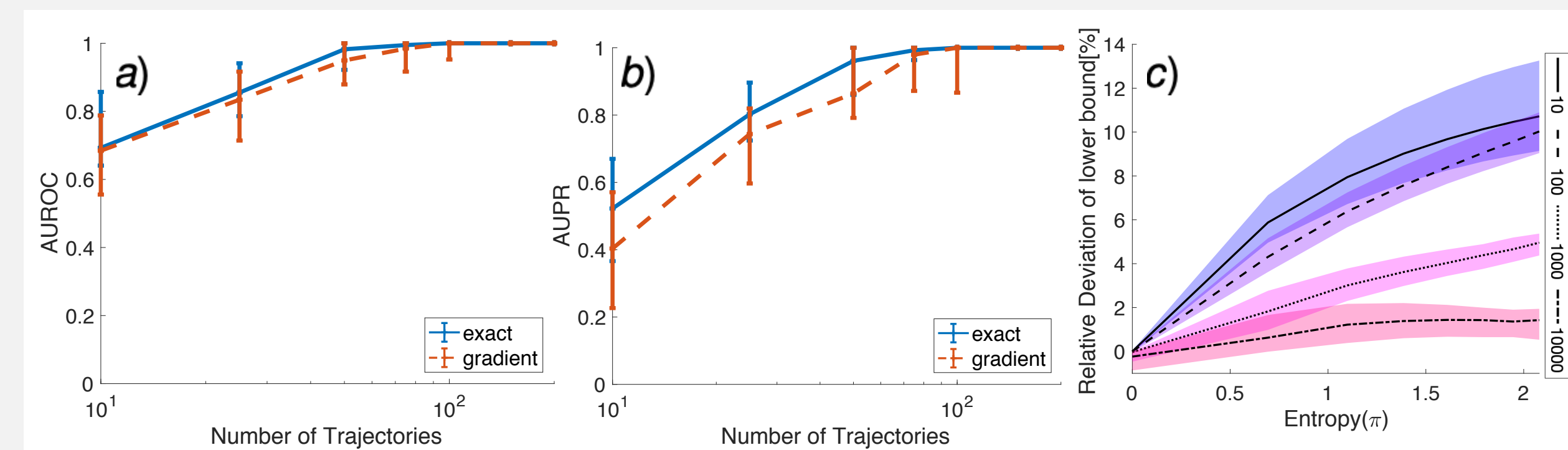


Figure 1: a) and b) AUROC and AUPR, respectively, for complete observations for different numbers of trajectories. Learning is performed via the graph-score (blue) and gradient-based optimization of the marginal mixture likelihood (red-dashed). c) Relative deviation of approximate marginal mixture likelihood from the exact marginal likelihood, computed via numerical integration, for mixtures of different entropies given different amounts of trajectories (legend). Confidence intervals are given by 75% and 25% percentiles.

Incomplete data

- For incomplete data \mathcal{Y} sufficient statistics are latent and need to be estimated \rightarrow intractable for large systems
- Solution: **Variational inference [1]**

$$p(X_{[0,T]} | \mathcal{Y}) \approx q(X_{[0,T]} | \mathcal{Y})$$

$$\mathcal{F}_i[\pi] \rightarrow \mathcal{F}_i[q, \pi], \quad M_i(x, x' | u_m) \rightarrow E_q[M_i(x, x' | u_m)], \quad T_i(x | u_m) \rightarrow E_q[T_i(x | u_m)]$$

- Yields an EM-Algorithm: $\pi_i^* = \arg \max_{\pi_i \in \Delta_i} \{\max_q \{\mathcal{F}_i[q, \pi]\}\}$
- Expectation can be performed efficiently via solving a set of linear ODEs:

$$\frac{d}{dt} \rho_i(t) = \tilde{\Omega}_i^\pi(t) \rho_i(t), \quad \frac{d}{dt} q_i(t) = q_i(t) \Omega_i^\pi(t),$$

Results

Synthetic data

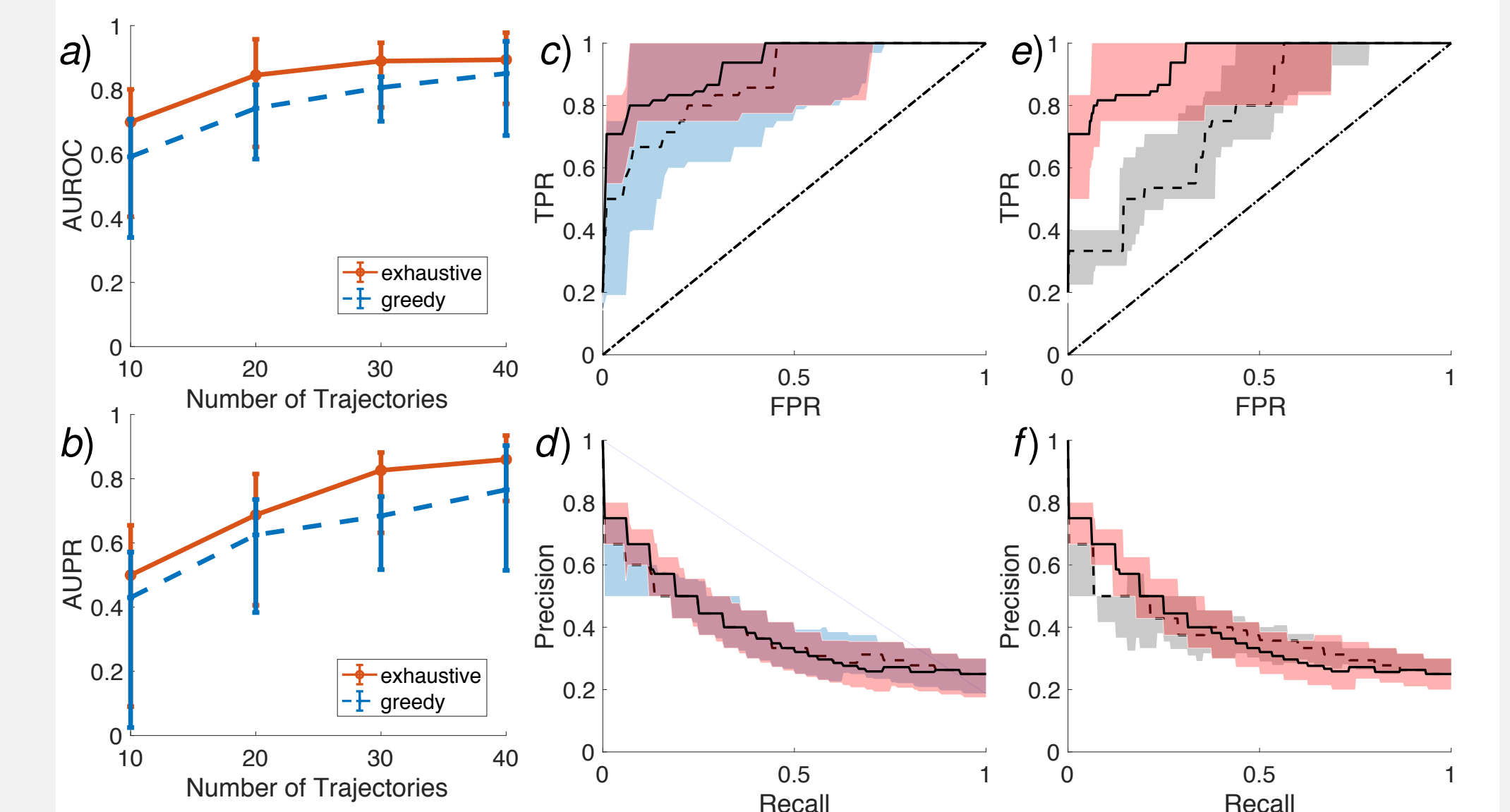


Figure 2: a) AUROCs and b) AUPRs for varying number of trajectories. c) ROC and d) PR curve for 40 trajectories. In all plots (red) denotes the exhaustive, (blue/dashed) the greedy-algorithm. e) ROC-curve f) PR-curve for different initials, where (red) denotes heuristic and (grey/dashed) random. Confidence intervals are given by 75% and 25% percentiles of the results from 30 random graphs, generated as explained in the main text.

Real-world data

- IRMA gene-regulatory network dataset [2]**
- Has been implemented on cultures of yeast for benchmarking

method		switch on	switch off
steady state	knockout	0.68 (0.42)	0.81 (0.50)
DBN	G1DBN	0.78 (0.64)	0.61 (0.34)
	VBSSM	0.79 (0.70)	0.76 (0.60)
ODE	TNSI	0.68 (0.51)	0.68 (0.42)
NDS	GP4GRN	0.73 (0.61)	0.76 (0.57)
	CSI ^d	0.63 (0.46)	0.86 (0.72)
	CSI ^c	0.64 (0.39)	0.73 (0.59)
GC	GCCA	0.71 (0.55)	0.74 (0.65)
CTBN	exhaustive	0.81 (0.86)	0.93 (0.92)
	greedy K=2	0.88 (0.85)	0.91 (0.89)
random		0.65 (0.45)	0.65 (0.45)

Table 1: AUROC (AUPR) of different methods on IRMA-data (top performers in bold).

- British Household dataset [3]**
- Time-course of questionnaires of british citizens about various facts about their lives

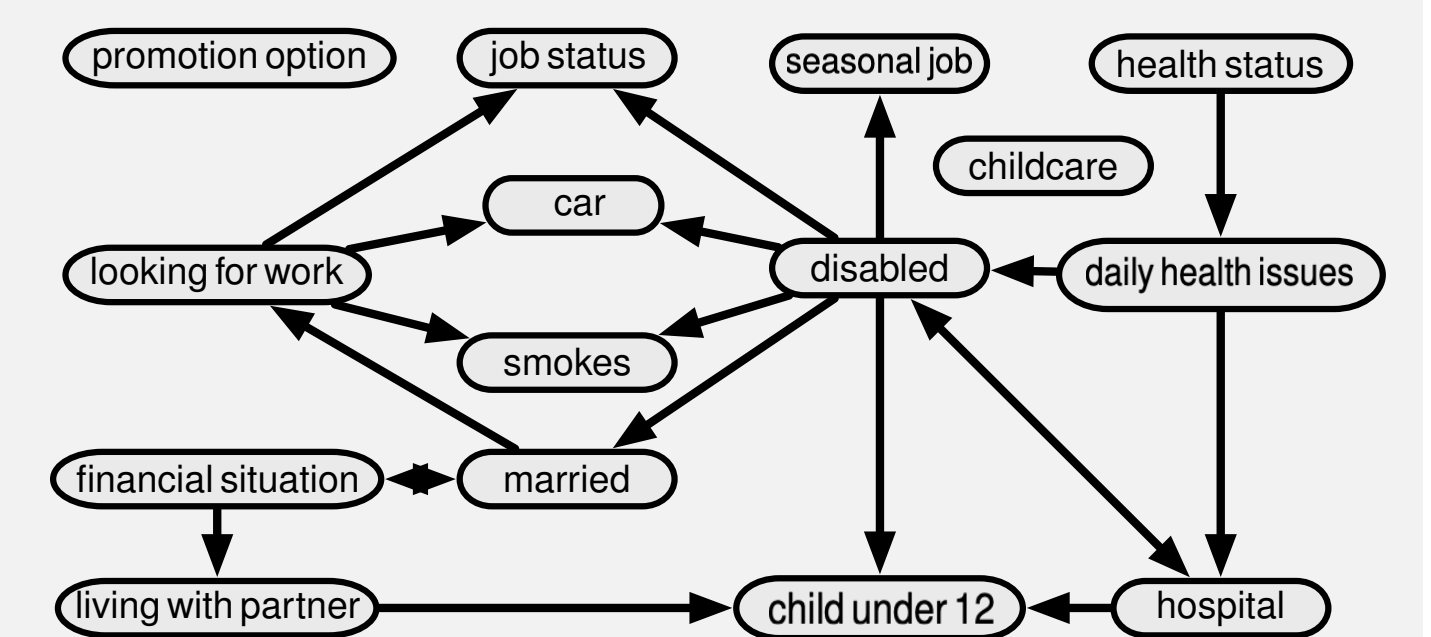


Figure 3: Learned structure using gradient-based greedy structure learning with maximal K=2 parents from 600 trajectories.

Acknowledgements

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- [3] British Household Panel Survey (ESRC Research Centre on Micro-social Change, 2003)