Chapter 7

MULTIMACHINE SIMULATION

In this chapter, we consider simulation techniques for a multimachine power system using a two-axis machine model with no saturation and neglecting both the stator and the network transients. The resulting differential-algebraic model is systematically derived. Both the partitioned-explicit (PE) and the simultaneous-implicit (SI) methods for integration are discussed. The SI method is preferred in both research grade programs and industry programs, since it can handle "stiff" equations very well. After explaining the SI method consistent with our analytical development so far, we then explain the equivalent but notationally different method, the well-known EPRI-ETMSP (Extended Transient Midterm Stability Program) [70]. A numerical example to illustrate the systematic computation of initial conditions is presented.

7.1 Differential-Algebraic Model

We first rewrite the two-axis model of Section 6.4 in a form suitable for simulation after neglecting the subtransient reactances and saturation. We also neglect the turbine governor dynamics resulting in T_{Mi} being a constant. The limit constraints on V_{Ri} are also deleted, since we wish to concentrate on modeling and simulation. We assume a linear damping term $T_{FWi} = D_i(\omega_i - \omega_s)$. The resulting differential-algebraic equations follow from (6.196)–(6.209) for the m machine, n bus system with the IEEE-Type I exciter as

1. Differential Equations

$$T'_{doi}\frac{dE'_{qi}}{dt} = -E'_{qi} - (X_{di} - X'_{di})I_{di} + E_{fdi} \qquad i = 1,\dots,m \tag{7.1}$$

$$T'_{qoi}\frac{dE'_{di}}{dt} = -E'_{di} + (X_{qi} - X'_{qi})I_{qi} \qquad i = 1,\dots,m \tag{7.2}$$

$$\frac{d\delta_i}{dt} = \omega_i - \omega_s \qquad i = 1,\dots,m \tag{7.3}$$

$$\begin{aligned}\frac{2H_i}{\omega_s}\frac{d\omega_i}{dt} &= T_{Mi} - E'_{di}I_{di} - E'_{qi}I_{qi} - (X'_{qi} - X'_{di})I_{di}I_{qi} \\ &\quad -D_i(\omega_i - \omega_s) \qquad i = 1,\dots,m\end{aligned} \tag{7.4}$$

$$T_{Ei}\frac{dE_{fdi}}{dt} = -(K_{Ei} + S_{Ei}(E_{fdi}))E_{fdi} + V_{Ri} \qquad i = 1,\dots,m \tag{7.5}$$

$$T_{Fi}\frac{dR_{fi}}{dt} = -R_{fi} + \frac{K_{Fi}}{T_{Fi}}E_{fdi} \qquad i = 1,\dots,m \tag{7.6}$$

$$\begin{aligned}T_{Ai}\frac{dV_{Ri}}{dt} &= -V_{Ri} + K_{Ai}R_{fi} - \frac{K_{Ai}K_{Fi}}{T_{Fi}}E_{fdi} + K_{Ai}(V_{\text{ref}_i} - V_i) \\ &\quad i = 1,\dots,m\end{aligned} \tag{7.7}$$

Equation (7.4) has dimensions of torque in per-unit. When the stator transients were neglected, the electrical torque became equal to the per-unit power associated with the internal voltage source.

2. Algebraic Equations

The algebraic equations consist of the stator algebraic equations and the network equations. The stator algebraic equations directly follow from the dynamic equivalent circuit of Figure 6.5, which is reproduced in Figure 7.1. Application of Kirchhoff's Voltage Law (KVL) to Figure 7.1 yields the stator algebraic equations:

(a) Stator algebraic equations

$$\begin{aligned}0 &= V_i e^{j\theta_i} + (R_{si} + jX'_{di})(I_{di} + jI_{qi})e^{j(\delta_i - \frac{\pi}{2})} \\ &\quad -[E'_{di} + (X'_{qi} - X'_{di})I_{qi} + jE'_{qi}]e^{j(\delta_i - \frac{\pi}{2})} \\ &\quad i = 1,\dots,m\end{aligned} \tag{7.8}$$

(b) Network equations

The dynamic circuit, together with the static network and the loads, is shown in Figure 7.2. The network equations written at the n buses are in complex form. From (6.208) and (6.209), these network equations are

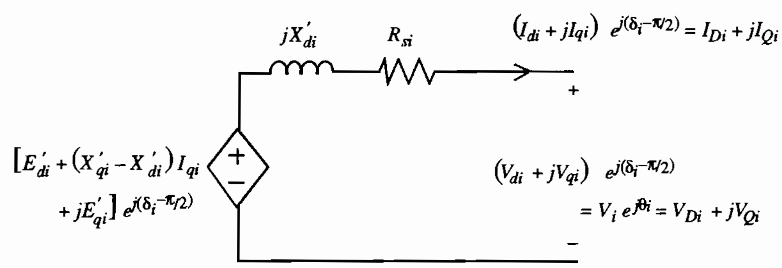

Figure 7.1: *Synchronous machine two-axis model dynamic circuit ($i = 1, \ldots, m$)*

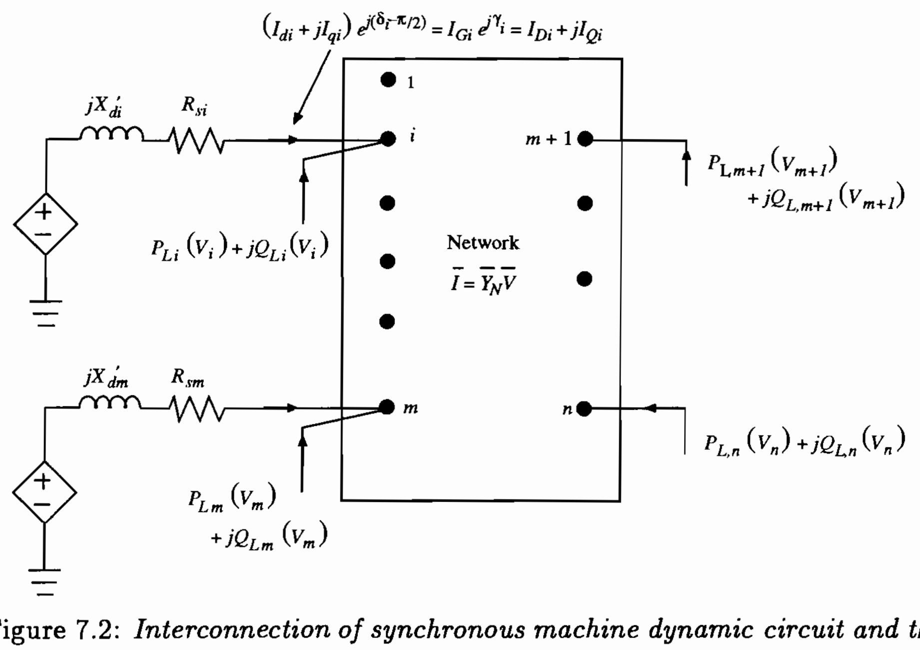

Figure 7.2: *Interconnection of synchronous machine dynamic circuit and the rest of the network*

Generator Buses

$$V_i e^{j\theta_i}(I_{di} - jI_{qi})e^{-j(\delta_i - \frac{\pi}{2})} + P_{Li}(V_i) + jQ_{Li}(V_i) = \sum_{k=1}^{n} V_i V_k Y_{ik} e^{j(\theta_i - \theta_k - \alpha_{ik})}$$

$$i = 1, \ldots, m \tag{7.9}$$

Load Buses

$$P_{Li}(V_i) + jQ_{Li}(V_i) = \sum_{k=1}^{n} V_i V_k Y_{ik} e^{j(\theta_i - \theta_k - \alpha_{ik})} \quad i = m+1, \ldots, n(7.10)$$

In (7.9), $V_i e^{j\theta_i}(I_{di} - jI_{qi})e^{-j(\delta_i - \pi/2)} \triangleq P_{Gi} + jQ_{Gi}$ is the complex power "injected" into bus i due to the generator. Thus, (7.9) and (7.10) are only the real and reactive power balance equation at all the n buses. Equation (7.9), which constitutes the power balance equations at the generator buses, shows the interaction of the algebraic variables and the state variables δ_i, E'_{qi}, and E'_{di}. We thus have

1. Seven differential equations (d.e.'s) for each machine, i.e., $7m$ d.e.'s ((7.1)–(7.7)).
2. One complex stator algebraic equation (7.8) (two real equations) for each machine, i.e., $2m$ real equations.
3. One complex network equation (7.9) and (7.10) (two real equations) at each network bus, i.e., $2n$ real equations.

We have $7m + 2m + 2n$ equations with $x = [x_1^t \ldots x_m^t]^t$ as the *state* vector where $x_i = [E'_{qi}\ E'_{di}\ \delta_i\ \omega_i\ E_{fdi}\ R_{fi}\ V_{Ri}]^t$ as the state vector for each machine. $y = [I_{d-q}^t\ V^t\ \theta^t]^t$ is the set of algebraic variables where

$$\begin{aligned} I_{d-q} &= [I_{d1}\ I_{q1} \ldots I_{dm} I_{qm}]^t \\ V &= [V_1 \ldots V_n]^t,\ \theta = [\theta_1 \ldots \theta_n]^t,\ \overline{V} = [\overline{V}_1 \ldots \overline{V}_n]^t \end{aligned}$$

Functionally, therefore, the differential equations (7.1)–(7.7), together with the stator algebraic equations (7.8) and the network equations (7.9)–(7.10), form a set of differential-algebraic equations of the form

$$\dot{x} = f(x, y, u) \tag{7.11}$$

$$0 = g(x, y) \tag{7.12}$$

$u = [u_1^t \ldots u_m^t]^t$ with $u_i = [\omega_s\ T_{mi}\ V_{\text{ref}_i}]^t$ as the input vector for each machine. We now formally put (7.1)–(7.10) in the form (7.11) and (7.12).

7.2 Stator Algebraic Equations

There are several different ways of writing the stator algebraic equations (7.8) as two real equations for computational purposes. The idea is to express I_{di}, I_{qi} in terms of the state and network variables. Both the polar form and the rectangular form will be explained.

7.2.1 Polar form

In this form, the network voltages appear in polar form. If we multiply (7.8) by $e^{-j(\delta_i - \frac{\pi}{2})}$ and equate the real and imaginary parts, we obtain

$$E'_{di} - V_i \sin(\delta_i - \theta_i) - R_{si} I_{di} + X'_{qi} I_{qi} = 0 \quad i = 1, \ldots, m \tag{7.13}$$

$$E'_{qi} - V_i \cos(\delta_i - \theta_i) - R_{si} I_{qi} - X'_{di} I_{di} = 0 \quad i = 1, \ldots, m \tag{7.14}$$

We define

$$\begin{bmatrix} R_{si} & -X'_{qi} \\ X'_{di} & R_{si} \end{bmatrix} \triangleq Z_{d-q,i}$$

Then, from (7.13) and (7.14):

$$\begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = [Z_{d-q,i}]^{-1} \begin{bmatrix} E'_{di} - V_i \sin(\delta_i - \theta_i) \\ E'_{qi} - V_i \cos(\delta_i - \theta_i) \end{bmatrix} \quad i = 1, \ldots, m \tag{7.15}$$

Equations (7.13) and (7.14) are implicit in I_{di}, I_{qi}, whereas in (7.15) they are expressed explicitly in terms of the state variables x_i and the algebraic variables V_i, θ_i. Thus

$$\begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = h_{pi}(x_i, V_i, \theta_i) \quad i = 1, \ldots, m \tag{7.16}$$

7.2.2 Rectangular form

This can be easily derived by recognizing the fact that

$$\overline{V}_i = V_{Di} + jV_{Qi} = V_i e^{j\theta_i} = V_i \cos\theta_i + jV_i \sin\theta_i \tag{7.17}$$

By expanding (7.13) and (7.14) and noting from (7.17) that $V_{Di} = V_i \cos\theta_i$ and $V_{Qi} = V_i \sin\theta_i$, we obtain the implicit form in rectangular coordinates as

$$E'_{di} - V_{Di} \sin\delta_i + V_{Qi} \cos\delta_i - R_{si} I_{di} + X'_{qi} I_{qi} = 0 \tag{7.18}$$

$$E'_{qi} - V_{Di} \cos\delta_i - V_{Qi} \sin\delta_i - R_{si} I_{qi} - X'_{di} I_{di} = 0 \tag{7.19}$$

To obtain the explicit form, I_{di}, I_{qi} in (7.18) and (7.19) can be expressed in terms of E'_{di}, E'_{qi}, δ_i, V_{Di}, and V_{Qi}. Alternatively, the right-hand side of (7.15) is expanded as

$$\begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = [Z_{d-q,i}]^{-1} \begin{bmatrix} E'_{di} \\ E'_{qi} \end{bmatrix} - [Z_{d-q,i}]^{-1} \begin{bmatrix} V_i(\sin\delta_i \cos\theta_i - \cos\delta_i \sin\theta_i) \\ V_i(\cos\delta_i \cos\theta_i + \sin\delta_i \sin\theta_i) \end{bmatrix} \tag{7.20}$$

Using the fact from (7.17) that $V_{Di} = V_i \cos\theta_i$ and $V_{Qi} = V_i \sin\theta_i$, (7.20) becomes

$$\begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = [Z_{d-q,i}]^{-1} \begin{bmatrix} E'_{di} \\ E'_{qi} \end{bmatrix} - [Z_{d-q,i}]^{-1} \begin{bmatrix} \sin\delta_i & -\cos\delta_i \\ \cos\delta_i & \sin\delta_i \end{bmatrix} \begin{bmatrix} V_{Di} \\ V_{Qi} \end{bmatrix} \tag{7.21}$$

$$= h_{ri}(x_i, V_{Di}, V_{Qi}) \quad i = 1, \ldots, m \tag{7.22}$$

Note that (7.21) can be obtained directly from (7.18) and (7.19). Symbolically, (7.16) or (7.22) can be expressed for all machines as

$$\begin{aligned} I_{d-q} &= h_p(x, V, \theta) \text{ or } h_r(x, V_D, V_Q) \\ &\triangleq h(x, \overline{V}) \end{aligned} \tag{7.23}$$

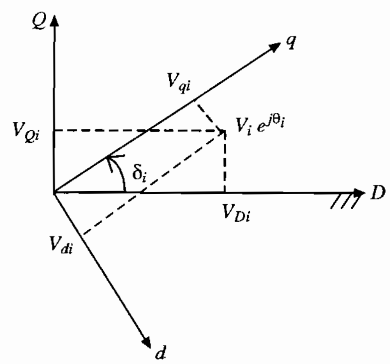

Figure 7.3: *Graphical representation*

7.2.3 Alternate form of stator algebraic equations

In much of the literature, a block diagram representation of stator equations is done through an "interface" block that reflects the machine–network transformation. The machine–network transformation is given by

$$\begin{bmatrix} F_{di} \\ F_{qi} \end{bmatrix} = \begin{bmatrix} \sin\delta_i & -\cos\delta_i \\ \cos\delta_i & \sin\delta_i \end{bmatrix} \begin{bmatrix} F_{Di} \\ F_{Qi} \end{bmatrix} \quad i = 1, \ldots, m \tag{7.24}$$

and

$$\begin{bmatrix} F_{Di} \\ F_{Qi} \end{bmatrix} = \begin{bmatrix} \sin\delta_i & \cos\delta_i \\ -\cos\delta_i & \sin\delta_i \end{bmatrix} \begin{bmatrix} F_{di} \\ F_{qi} \end{bmatrix} \quad i = 1, \ldots, m \tag{7.25}$$

where F may be either I or V. Figure 7.3 is a graphical representation of (7.24) and (7.25) illustrated for the voltage $\overline{V}_i = V_i e^{j\theta_i}$. Using (7.24) in (7.21), we obtain

$$\begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = [Z_{d-q,i}]^{-1} \begin{bmatrix} E'_{di} - V_{di} \\ E'_{qi} - V_{qi} \end{bmatrix} \quad i = 1, \ldots, m \tag{7.26}$$

Thus

$$\begin{bmatrix} E'_{di} - V_{di} \\ E'_{qi} - V_{qi} \end{bmatrix} = [Z_{d-q,i}] \begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} \quad i = 1, \ldots, m \tag{7.27}$$

The interface block in the block diagram in Figure 7.5 is now consistent with (7.24), (7.25), and (7.26). Note that, in this formulation, algebraic equation (7.26) or (7.27) is in machine reference only, whereas (7.24) and (7.25) act as an "interface" between the machine and the network.

7.3 Network Equations

The network equations can be expressed either in power-balance or current-balance form. The latter form is more popular with the industry software packages. We discuss both of them now.

7.3.1 Power-balance form

The network equations for the generator buses ((7.9)) are separated into real and imaginary parts for $i = 1, \ldots, m$

$$I_{di}V_i \sin(\delta_i - \theta_i) + I_{qi}V_i \cos(\delta_i - \theta_i) + P_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k - \alpha_{ik}) = 0 \tag{7.28}$$

$$I_{di}V_i \cos(\delta_i - \theta_i) - I_{qi}V_i \sin(\delta_i - \theta_i) + Q_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k - \alpha_{ik}) = 0 \tag{7.29}$$

For the load buses, a similiar procedure using (7.10) gives for $i = m+1, \ldots, n$

$$P_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k - \alpha_{ik}) = 0 \tag{7.30}$$

$$Q_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k - \alpha_{ik}) = 0 \tag{7.31}$$

Note that load can be present at the generator as well as at the load buses. The network equations (7.28)–(7.31) can be rearranged so that the real power equations appear first and the reactive power equations appear next, as follows.

Real Power Equations

$$I_{di}V_i \sin(\delta_i - \theta_i) + I_{qi}V_i \cos(\delta_i - \theta_i) + P_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k - \alpha_{ik}) = 0 \quad i = 1, \ldots, m \tag{7.32}$$

$$P_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \cos(\theta_i - \theta_k - \alpha_{ik}) = 0 \quad i = m+1, \ldots, n \tag{7.33}$$

Reactive Power Equations

$$I_{di}V_i \cos(\delta_i - \theta_i) - I_{qi}V_i \sin(\delta_i - \theta_i) + Q_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k - \alpha_{ik}) = 0 \tag{7.34}$$

$$Q_{Li}(V_i) - \sum_{k=1}^{n} V_i V_k Y_{ik} \sin(\theta_i - \theta_k - \alpha_{ik}) = 0 \quad i = m+1, \ldots, n \tag{7.35}$$

Thus, the differential-algebraic equation (DAE) model is:

1. The differential equations (7.1)–(7.7)
2. The stator algebraic equations of the form (7.23) in the polar form
3. The network equations (7.32)–(7.35) in the power-balance form

The differential-algebraic equations are now written symbolically as

$$\dot{x} = f_o(x, I_{d-q}, \overline{V}, u) \tag{7.36}$$

$$I_{d-q} = h(x, \overline{V}) \tag{7.37}$$

$$0 = g_o(x, I_{d-q}, \overline{V}) \tag{7.38}$$

Substitution of (7.37) into (7.36) and (7.38) gives

$$\dot{x} = f_1(x, \overline{V}, u) \tag{7.39}$$

$$0 = g_1(x, \overline{V}) \tag{7.40}$$

Note that (7.40) is in the power-balance form. This is the differential-algebraic equation (DAE) analytical model with the network algebraic variables in the polar form. We prefer this form, since in load-flow equations the voltages are generally in polar form. Simplified forms of this model result from the reduced-order model of the synchronous machine as well as the exciter, which will be discussed later.

7.3.2 Current-balance form

Instead of the power-balance form of (7.32)–(7.35), one can have the current-balance form, which is essentially the nodal set of equations

$$\overline{I} = \overline{Y}_N \overline{V} \tag{7.41}$$

where $\overline{Y}_N$ is the $n \times n$ bus admittance matrix of the network with elements $\overline{Y}_{ik} = Y_{ik} e^{j\alpha_{ik}} = G_{ik} + jB_{ik}$, $\overline{I}$ is the net injected current vector and $\overline{V}$ is the bus voltage vector. Depending on how $\overline{I}$ is expressed, it can take different forms, as discussed below. Equation (7.41) can also be derived from (7.9)–(7.10) by dividing both sides of the equation by $V_i e^{j\theta_i}$ and then taking the complex conjugate as follows.

$$(I_{di} + jI_{qi})e^{j(\delta_i - \pi/2)} + \frac{P_{Li}(V_i) - jQ_{Li}(V_i)}{V_i e^{-j\theta_i}} = \sum_{k=1}^{n} Y_{ik} e^{j\alpha_{ik}} V_k e^{j\theta_k} \qquad i = 1, \ldots, m \tag{7.42}$$

$$\frac{P_{Li}(V_i) - jQ_{Li}(V_i)}{V_i e^{-j\theta_i}} = \sum_{k=1}^{n} Y_{ik} e^{j\alpha_{ik}} V_k e^{j\theta_k} \quad i = m+1, \ldots, n \tag{7.43}$$

These equations are the same as (6.83) and (6.84). Equations (7.42) and (7.43) can be symbolically denoted in matrix form as

$$\overline{I}_o(I_{d-q}, x, \overline{V}) = \overline{Y}_N \overline{V} \tag{7.44}$$

The other algebraic equation is

$$I_{d-q} = h(x, \overline{V}) \tag{7.45}$$

Substitution of (7.45) in (7.36) and (7.44) leads to the DAE model

$$\begin{aligned} \dot{x} &= f_1(x, \overline{V}, u) \\ \overline{I}_1(x, \overline{V}) &= \overline{Y}_N \overline{V} \end{aligned} \tag{7.46}$$

Example 7.1

We illustrate the DAE models discussed in the previous section with a numerical example. We consider the popular Western System Coordinating Council (WSCC) 3-machine, 9-bus system [73] shown in Figure 7.4. This is also the system appearing in [74] and widely used in the literature. The base MVA is 100, and system frequency is 60 Hz. The converged load-flow data obtained using the EPRI-IPFLOW program [75] is given in Table 7.1.

The $\overline{Y}_{\text{bus}}$ for the network (also denoted as $\overline{Y}_N$) can be written by inspection from Figure 7.4 and is shown in Table 7.2. The machine data and the exciter data are given in Table 7.3. The exciter is assumed to be identical for all the machines and is of the IEEE-Type I. Define $\frac{2H_i}{\omega_s} \triangleq M_i$. Assume that $\frac{D_1}{M_1} = 0.1$, $\frac{D_2}{M_2} = 0.2$, and $\frac{D_3}{M_3} = 0.3$ (all in pu).

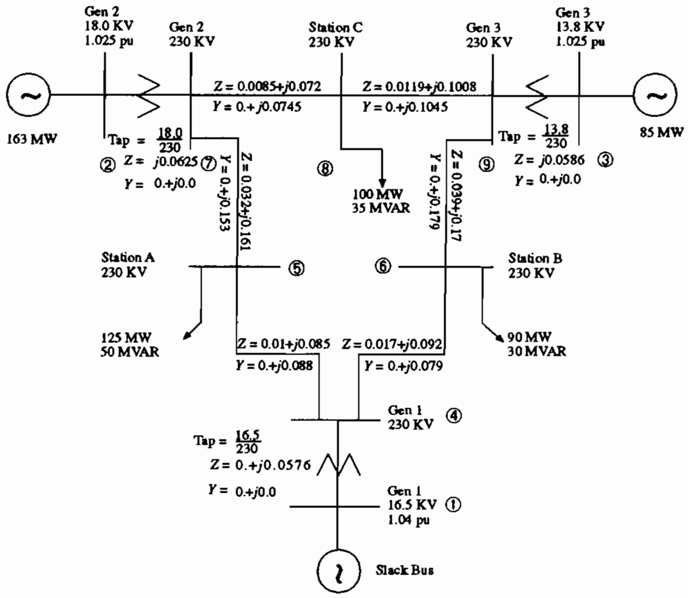

Figure 7.4: *WSCC 3-machine, 9-bus system; the value of Y is half the line charging (Copyright 1977. Electric Power Research Institute. EPRI EL-0484. Power System Dynamic Analysis, Phase I. Reprinted with Permission.).*

Table 7.1: Load-Flow Results of the WSCC 3-Machine, 9-Bus System

	Bus #	Voltage (pu)	P_G (pu)	Q_G (pu)	$-P_L$ (pu)	$-Q_L$ (pu)
1	(swing)	1.04	0.716	0.27	–	–
2	(P-V)	$1.025\angle 9.3^o$	1.63	0.067	–	–
3	(P-V)	$1.025\angle 4.7^o$	0.85	-0.109	–	–
4	(P-Q)	$1.026\angle -2.2^o$	–	–	–	–
5	(")	$0.996\angle$-4.0^o	–	–	1.25	0.5
6	(")	$1.013\angle -3.7^o$	–	–	0.9	0.3
7	(")	$1.026\angle 3.7^o$	–	–	–	–
8	(")	$1.016\angle 0.7^o$	–	–	1.00	0.35
9	(")	$1.032\angle 2.0^o$	–	–	–	–

Table 7.2: $\overline{Y}_N$ for the Network in Figure 7.4

	1	2	3	4	5	6	7	8	9
1	$-j17.361$	0	0	$j17.361$	0	0	0	0	0
2	0	$-j16$	0	0	0	0	$j16$	0	0
3	0	0	$-j17.065$	0	0	0	0	0	$j17.065$
4	$j17.361$	0	0	3.307 $-j39.309$	-1.365 $+j11.604$	-1.942 $+j10.511$	0	0	0
5	0	0	0	-1.365 $+j11.604$	2.553 $-j17.338$	0	-1.188 $+j5.975$	0	0
6	0	0	0	-1.942 $+j10.511$	0	3.224 $-j15.841$	0	0	-1.282 $+j5.588$
7	0	$j16$	0	0	-1.188 $+j5.975$	0	2.805 $-j35.4460$	-1.617 $+j13.698$	0
8	0	0	0	0	0	0	-1.617 $+j13.698$	2.772 $-j23.303$	-1.155 $+j9.784$
9	0	0	$j17.065$	0	0	-1.282 $+j5.588$	0	-1.155 $+j9.784$	2.437 $-j32.1540$

Table 7.3: Machine and Exciter Data

Machine Data

Parameters	M/C 1	M/C 2	M/C 3
H(secs)	23.64	6.4	3.01
X_d(pu)	0.146	0.8958	1.3125
X'_d(pu)	0.0608	0.1198	0.1813
X_q(pu)	0.0969	0.8645	1.2578
X'_q(pu)	0.0969	0.1969	0.25
T'_{do}(sec)	8.96	6.0	5.89
T'_{qo}(sec)	0.31	0.535	0.6

Exciter Data

Parameters	Exciter 1	Exciter 2	Exciter 3
K_A	20	20	20
T_A(sec)	0.2	0.2	0.2
K_E	1.0	1.0	1.0
T_E(sec)	0.314	0.314	0.314
K_F	0.063	0.063	0.063
T_F(sec)	0.35	0.35	0.35

$$S_{Ei}(E_{fdi}) = 0.0039e^{1.555E_{fdi}} \qquad i = 1, 2, 3$$

The differential equations corresponding to (7.1)–(7.7) are

$$\begin{bmatrix} \dot{E}'_{qi} \\ \dot{E}'_{di} \\ \dot{\delta}_i \\ \dot{\omega}_i \\ \dot{E}_{fdi} \\ \dot{R}_{fi} \\ \dot{V}_{Ri} \end{bmatrix} = [A_i] \begin{bmatrix} E'_{qi} \\ E'_{di} \\ \delta_i \\ \omega_i \\ E_{fdi} \\ R_{fi} \\ V_{Ri} \end{bmatrix} + R_i(E'_{qi}, E'_{di}, E_{fdi}, I_{di}, I_{qi}, V_i) + C_i u_i \qquad i = 1,2,3 \tag{7.47}$$

where

$$A_i = \begin{bmatrix} \frac{-1}{T'_{doi}} & 0 & 0 & 0 & \frac{1}{T'_{doi}} & 0 & 0 \\ 0 & \frac{-1}{T'_{qoi}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-D_i}{M_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_{Ei}}{T_{Ei}} & 0 & \frac{1}{T_{Ei}} \\ 0 & 0 & 0 & 0 & \frac{K_{Fi}}{T_{Fi}^2} & \frac{-1}{T_{Fi}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-K_{Ai}K_{Fi}}{T_{Ai}T_{Fi}} & \frac{K_{Ai}}{T_{Ai}} & \frac{-1}{T_{Ai}} \end{bmatrix} \qquad i = 1,2,3 \tag{7.48}$$

$$R_i = \begin{bmatrix} \frac{-(X_{di}-X'_{di})I_{di}}{T'_{doi}} \\ \frac{(X_{qi}-X'_{qi})I_{qi}}{T'_{qoi}} \\ 0 \\ \frac{-\omega_s}{2H_i}[(E'_{di}I_{di} + E'_{qi}I_{qi}) + (X'_{qi} - X'_{di})I_{di}I_{qi}] \\ -\frac{S_{Ei}(E_{fdi})}{T_{Ei}} \\ 0 \\ \frac{-K_{Ai}}{T_{Ai}}V_i \end{bmatrix}$$

$$C_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ \frac{D_i}{M_i} & \frac{1}{M_i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{K_{Ai}}{T_{Ai}} \end{bmatrix} \quad u_i = \begin{bmatrix} \omega_s \\ T_{Mi} \\ V_{\text{ref}i} \end{bmatrix} \quad i = 1, 2, 3 \qquad (7.49)$$

Substituting the numerical values, we obtain

$$A_1 = \begin{bmatrix} -0.112 & 0 & 0 & 0 & 0.112 & 0 & 0 \\ 0 & -3.226 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.185 & 0 & 3.185 \\ 0 & 0 & 0 & 0 & 0.514 & -2.86 & 0 \\ 0 & 0 & 0 & 0 & -18 & 100 & -5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.167 & 0 & 0 & 0 & 0.167 & 0 & 0 \\ 0 & -1.87 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.185 & 0 & 3.185 \\ 0 & 0 & 0 & 0 & 0.514 & -2.86 & 0 \\ 0 & 0 & 0 & 0 & -18 & 100 & -5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.17 & 0 & 0 & 0 & 0.17 & 0 & 0 \\ 0 & -1.67 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.185 & 0 & 3.185 \\ 0 & 0 & 0 & 0 & 0.514 & -2.86 & 0 \\ 0 & 0 & 0 & 0 & -18 & 100 & -5 \end{bmatrix} \qquad (7.50)$$

$$R_1 = \begin{bmatrix} -0.0095I_{d1} \\ 0 \\ 0 \\ -8(E'_{d1}I_{d1} + E'_{q1}I_{q1}) \\ -0.29I_{d1}I_{q1} \\ 0.0124e^{1.555E_{fd1}} \\ 0 \\ -100V_1 \end{bmatrix}, \; R_2 = \begin{bmatrix} -0.13I_{d2} \\ 1.25I_{q2} \\ 0 \\ -29.5(E'_{d2}I_{d2} + E'_{q2}I_{q2}) \\ -2.27I_{d2}I_{q2} \\ 0.0124e^{1.555E_{fd2}} \\ 0 \\ -100V_2 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} -0.19I_{d3} \\ 1.7I_{q3} \\ 0 \\ -62.6(E'_{d3}I_{d3} + E'_{q3}I_{q3}) \\ -4.3I_{d3}I_{q3} \\ 0.0124e^{1.555E_{fd3}} \\ 0 \\ -100V_3 \end{bmatrix} \tag{7.51}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0.1 & 8 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0.2 & 29.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0.3 & 62.6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix}. \tag{7.52}$$

The stator algebraic equations corresponding to (7.13) and (7.14) (assuming $R_{si} \equiv 0$) are

$$E'_{d1} - V_1 \sin(\delta_1 - \theta_1) + 0.0969I_{q1} = 0$$

$$\begin{aligned}
E'_{q1} - V_1 \cos(\delta_1 - \theta_1) - 0.0608 I_{d1} &= 0 \\
E'_{d2} - V_2 \sin(\delta_2 - \theta_2) + 0.1969 I_{q2} &= 0 \\
E'_{q2} - V_2 \cos(\delta_2 - \theta_2) - 0.1198 I_{d2} &= 0 \\
E'_{d3} - V_3 \sin(\delta_3 - \theta_3) + 0.2500 I_{q3} &= 0 \\
E'_{q3} - V_3 \cos(\delta_3 - \theta_3) - 0.1813 I_{d3} &= 0 \qquad (7.53)
\end{aligned}$$

The network equations are (with the notation $\theta_{ij} = \theta_i - \theta_j$) as follows. The constant power loads are treated as injected into the buses.

Real Power Equations

$$\begin{aligned}
I_{d1} V_1 \sin(\delta_1 - \theta_1) &+ I_{q1} V_1 \cos(\delta_1 - \theta_1) \\
&- 17.36 V_1 V_4 \sin \theta_{14} = 0 \\
I_{d2} V_2 \sin(\delta_2 - \theta_2) &+ I_{q2} V_2 \cos(\delta_2 - \theta_2) \\
&- 16.00\, V_2 V_7 \sin \theta_{27} = 0 \\
I_{d3} V_3 \sin(\delta_3 - \theta_3) &+ I_{q3} V_3 \cos(\delta_3 - \theta_3) \\
&- 17.06 V_3 V_9 \sin \theta_{39} = 0 \\
-17.36 V_4 V_1 \sin \theta_{41} &- 3.31 V_4^2 + 1.36 V_4 V_5 \cos \theta_{45} - 11.6 V_4 V_5 \sin \theta_{45} \\
&+ 1.942 V_4 V_6 \cos \theta_{46} - 10.51 V_4 V_6 \sin \theta_{46} = 0 \\
-1.25 + 1.36 V_5 V_4 \cos \theta_{54} &- 11.6 V_5 V_4 \sin \theta_{54} + 1.19 V_5 V_7 \cos \theta_{57} \\
&- 5.97 V_5 V_7 \sin \theta_{57} - 2.55 V_5^2 = 0 \\
-0.9 + 1.94 V_6 V_4 \cos \theta_{64} &- 10.51 V_6 V_4 \sin \theta_{64} - 3.22 V_6^2 \\
&+ 1.28 V_6 V_9 \cos \theta_{69} - 5.59 V_6 V_9 \sin \theta_{69} = 0 \\
-16 V_7 V_2 \sin \theta_{72} &+ 1.19 V_7 V_5 \cos \theta_{75} - 5.98 V_7 V_5 \sin \theta_{75} \\
&- 2.8 V_7^2 + 1.62 V_7 V_8 \cos \theta_{78} \\
&- 13.7 V_7 V_8 \sin \theta_{78} = 0 \\
-1 + 1.62 V_8 V_7 \cos \theta_{87} &- 13.7 V_8 V_7 \sin \theta_{87} - 2.77 V_8^2 \\
&+ 1.16 V_8 V_9 \cos \theta_{89} - 9.8 V_8 V_9 \sin \theta_{89} = 0 \\
-17.065 V_9 V_3 \sin \theta_{93} &+ 1.28 V_9 V_6 \cos \theta_{96} - 5.59 V_9 V_6 \sin \theta_{96}
\end{aligned}$$

$$+ 1.15V_9V_8\cos\theta_{98} - 9.78V_9V_8\sin\theta_{98}$$
$$- 2.4V_9^2 = 0 \qquad (7.54)$$

Reactive Power Equations

$$\begin{aligned}
I_{d1}V_1\cos(\delta_1-\theta_1) &- I_{q1}V_1\sin(\delta_1-\theta_1)\\
&+ 17.36V_1V_4\cos\theta_{14} - 17.36V_1^2 = 0\\
I_{d2}V_2\cos(\delta_2-\theta_2) &- I_{q2}V_2\sin(\delta_2-\theta_2)\\
&+ 16V_2V_7\cos\theta_{27} - 16V_2^2 = 0\\
I_{d3}V_3\cos(\delta_3-\theta_3) &- I_{q3}V_3\sin(\delta_3-\theta_3)\\
&+ 17.07V_3V_9\cos\theta_{39} - 17.07V_3^2 = 0\\
17.36V_4V_1\cos\theta_{41} &- 39.3V_4^2 + 1.36V_4V_5\sin\theta_{45}\\
&+ 11.6V_4V_5\cos\theta_{45} + 1.94V_4V_6\sin\theta_{46}\\
&+ 10.52V_4V_6\cos\theta_{46} = 0\\
-0.5 + 1.37V_5V_4\sin\theta_{54} &+ 11.6V_5V_4\cos\theta_{54} - 17.34V_5^2\\
&+ 1.19V_5V_7\sin\theta_{57} + 5.98V_5V_7\cos\theta_{57} = 0\\
-0.3 + 1.94V_6V_4\sin\theta_{64} &+ 10.51V_6V_4\cos\theta_{64} - 15.84V_6^2\\
&+ 1.28V_6V_9\sin\theta_{69} + 5.59V_6V_9\cos\theta_{69} = 0\\
16V_7V_2\cos\theta_{72} &+ 1.19V_7V_5\sin\theta_{75} + 5.98V_7V_5\cos\theta_{75}\\
&- 35.45V_7^2 + 1.62V_7V_8\sin\theta_{78} + 13.67V_7V_8\cos\theta_{78} = 0\\
-0.35 + 1.62V_8V_7\sin\theta_{87} &+ 13.67V_8V_7\cos\theta_{87} - 23.3V_8^2 + 1.15V_8V_9\sin\theta_{89}\\
&+ 9.78V_8V_9\cos\theta_{89} = 0\\
17.065V_9V_3\cos\theta_{93} &+ 1.28V_9V_6\sin\theta_{96} + 5.59V_9V_6\cos\theta_{96}\\
&+ 1.16V_9V_8\sin\theta_{98} + 9.78V_9V_8\cos\theta_{98} - 32.15V_9^2 = 0
\end{aligned} \qquad (7.55)$$

It is easy to solve (7.53) for I_{di}, I_{qi} $(i = 1,2,3)$, substitute them in (7.47) and (7.54)–(7.55), and obtain the equations $\dot{x} = f_1(x, \overline{V}, u)$ and $0 = g_1(x, \overline{V})$. This is left as an exercise for the reader.

□