Modeling Traffic Jams in Extracellular Transport in Axons

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Abstract

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1 Introduction

Traffic jams occur in our everyday life. Most people use the road, either by car, bus or bike, every day to get to their job, do grocery shopping, to meet friends or to get to their hobbies. Through this enormous use of the road, traffic jams occur. Traffic jams lead to a smaller speed of vehicles, if too many vehicles are close to each other.

The arising engineering question is, how to reduce these traffic jams? With a reduction of traffic jams, the traffic quality can increase, with less time on the road. However, to reduce traffic jams it is essential to understand how these work.

For this, it is necessary to study the underlying mathematical model of traffic jams, which states a smaller vehicle speed at higher number densities.

1.1 The Mathematical Model

The underlying mathematical model of traffic jams relies on the general conservation of mass

$$u_t + (V(u)u)_x = 0, \quad x \in [a, b] \tag{1.1}$$

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Figure 1.1. Image by Al Gr from Pixabay

where u is the number density of vehicles ($u \in [a, b]$), $V(u)$ is the velocity of vehicles, depending on the number density and $b - a$ is the length of the road. Furthermore, an inflow boundary condition

$$u\big|_{x=a} = u_{\text{in}} \tag{1.2}$$

is assumed to hold true. This inflow boundary condition models the number of arriving cars at the beginning of the road.

Finally, the problem can be reformulated to the general, scalar, nonlinear conservation law

$$u_t + f(u)_x = 0 \quad \text{in} x \in \Omega \tag{1.3}$$
$$u\big|_{x=x_{\text{in}}} = u_{\text{in}} \tag{1.4}$$

The main assumption for this model is, that the vehicle velocity depends on the number density

$$V(u) \propto (1 - u) \tag{1.5}$$

Increasing number densities lead to a decreased vehicle velocity and vice versa. Finally, the velocity is scaled to its maximum value V_{max} and this maximum value is set to 1 ($V_{\text{max}} = 1$), resulting in

$$V(u) = V_{\text{max}}(1 - u) \tag{1.6}$$
$$= (1 - u) \tag{1.7}$$
$$\Rightarrow u_t + (u(1 - u))_x = 0, \quad x \in [a, b] \tag{1.8}$$
$$\Leftrightarrow u_t + (f(u))_x = 0 \tag{1.9}$$
$$\text{for: } f(u) = u(1 - u) \tag{1.10}$$

1.2 Arising Research Questions

From this mathematical model and the physical background some research questions arise.

1) How does the number density evolve over time? What influence has the initial distribution on the transient behavior of the number densities?
2) What is the highest number density? Is there a mathematical explanation for this highest number density?
3) What is the flux of the moving vehicles? How does it change over time? When is it high and when is it low?
4) Is it more efficient to have a high distance or short distance between the cars? With a higher distance a higher speed is possible, but less cars are on the road.

2 Theory and Methods

General nonlinear conservation law

$$u_t + f(u)_x = 0, \quad u(x, t = 0) = u_0(x) \tag{2.1}$$

2.1 General Solution for the Nonlinear Conservation Law

First, note that $f(u)_x = f'(u)u_x$ holds true. With this, the problem can be reformulated to

$$\begin{cases} u_t + f'(u)u_x \\ u(x, t = 0) = u_0(x) \end{cases} \tag{2.2}$$

As a first step, the time derivative of $u(x(t), t)$ will be calculated

$$\frac{d}{dt}u(x(t), t) = u_t \cancel{\frac{dt}{dt}} + u_x \frac{dx}{dt} = u_t + u_x f'(u) \overset{!}{=} 0 \tag{2.3}$$

from this it can be concluded that $u(x(t), t)$ has to be constant ($u(x(t), t) = \text{const} = u_0(x_0)$). With this knowledge, the characteristic for $x(t)$ can be determined. By comparing equation 2.3 with the general equation, the ODE

$$\frac{dx}{dt}(t) = f'\left(u(x(t), t)\right), \quad x(t = 0) = x_0 \tag{2.4}$$
$$= f'\left(u_0(x_0)\right) \tag{2.5}$$

concludes, which can be solved by separation of variables

$$dx = f'\left(u_0(x_0)\right) dt \tag{2.6}$$
$$\Leftrightarrow \int_{x(t=0)}^{x(t)} 1 dx = \int_0^1 f'\left(u_0(x_0)\right) dt = f'\left(u_0(x_0)\right) t \tag{2.7}$$
$$\Rightarrow x(t) = x_0 + f'\left(u_0(x_0)\right) t \tag{2.8}$$

Inserting the new expression for $x(t)$ into equation 2.2 results in the final analytical solution

$$\begin{aligned} u(x,t) &= u_0(x_0) && (2.9)\\ &= u_0\left(x(t) - f'((u_0(x_0))\,t\right) && (2.10)\\ &= u_0\left(x - f'\left(u(x,t)\right)t\right) && (2.11) \end{aligned}$$

A natural arising question is, if this solution holds true for all possible functions of f. The initial condition is always fulfilled with $u(x,t=0) = u_0(x)$. However, the PDE is only fulfilled under the constraint $1 + u_0'(x - f'(u(x,t))t) + f''(u(x,t)) \neq 0$.

Proof: $1 + u_0'(x - f'(u(x,t))t) + f''(u(x,t)) \neq 0$ has to be fulfilled to transform $u(x,t)$ into a solution

First, calculate the partial derivatives of u.

$$u_t = \frac{\partial}{\partial t}\left(u_0(\underbrace{x - f'\left(u(x,t)\right)t}_{v(x,t)}\right) = \frac{du_0}{dv}\frac{\partial v}{\partial t} \tag{2.12}$$

$$\begin{aligned} \text{with: } \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} = \left(x - f'\left(u(x,t)\right)t\right) = -\left(f'(u) + tf''(u)u_t\right) && (2.13)\\ &= -u_0'\left(v(x,t)\right)\left(f'(u) + tf''(u)u_t\right) && (2.14)\\ u_x &= \frac{\partial}{\partial x}\left(u_0(v(x,t))\right) = \frac{du_0}{dv}\frac{\partial v}{\partial x} && (2.15)\\ \text{with: } \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left(x - f'(u)t\right) = 1 - f''(u)tu_x && (2.16)\\ &= u_0\left(v(x,t)\right)\left(1 - tf''(u)u_x\right) && (2.17) \end{aligned}$$

$$\Rightarrow f(u)_x = f'(u)u_x = u_0'\left(v(x,t)\right)\left(f'(u) - tf''(u)\underbrace{u_x f'(u)}_{f(u)_x}\right) \tag{2.18}$$

Inserting the expressions for u_t and $f(u)_x$ into the PDE results in

$$\begin{aligned} u_t + f(u)_x &= -u_0'(v)\left(\cancel{f'(u)} + tf''(u)u_t\right) + u_0'(v)\left(\cancel{f'(u)} - tf''(u)f(u)_x\right) && (2.19)\\ &= -u_0'(v)tf''(u)\left(u_t + f(u)u_x\right) && (2.20)\\ \Leftrightarrow \left(u_t + f(u)_x\right)\left(1 + u_0'(v)t + f''(u)\right) &= 0 && (2.21) \end{aligned}$$

Therefore, the condition $1 + u_0'(x - f'(u(x,t))t) + f''(u(x,t)) \neq 0$ is sufficient to guarantee, that $u(x,t)$ fulfills the PDE. However, if the expression is zero, it cannot be guaranteed that the PDE is fulfilled. □

2.2 Formation of Discontinuities

An advection equation may form discontinuities, also known as shocks for $u_x \notin \mathcal{R}$. Therefore, shocks are formed for $1 + u_0'(x_0)f''(u_0(x_0))t = 0$

Proof: Shocks are formed for $1 + u_0'(x_0)f''(u_0(x_0))t = 0$

Consider the solution for the general conservation law

$$u(x,t) = u_0\left(\underbrace{x - f'(u(x,t)t}_{v(x,t)}\right) \tag{2.22}$$

and calculate the spatial derivative (u_x) of this

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}u_0(v) = u_0'(v)\frac{\partial v}{\partial x} = u_0'(v)\frac{\partial}{\partial x}\left(x - f'(u)t\right) && (2.23)\\ &= u_0'(v)\left(1 - f''(u)u_x t\right) && (2.24)\\ \Leftrightarrow u_x\left(1 + u_0'(v)f''(u)t\right) &= u_0'(v) && (2.25)\\ u_x(x,t) &= \frac{u_0'(v)}{1 + u_0'(x)f''\left(u_0(x_0)\right)t} && (2.26) \end{aligned}$$

Conclude from this

$$u(x,t) \text{ is continuous } \forall u_x \in \mathcal{R}$$
$$u(x,t) \text{ is discontinuous if } 1 + u_0'(x)f''\left(u_0(x_0)\right)t = 0$$

□

2.2.1 Formation of Discontinuities for the case $f(u) = u^2$

Consider the convex, scalar function $f(u) = u^2$ with the derivatives

$$f'(u) = 2u \quad f''(u) = 2$$

And the initial condition

$$u_0(x) = \begin{cases} 2x & ,0 \leq x \leq \frac{1}{2} \\ 2(1-x) & ,\frac{1}{2} < x \leq 1 \end{cases} \qquad u_0'(x) = \begin{cases} 2 & ,0 \leq x \leq \frac{1}{2} \\ -2 & ,\frac{1}{2} < x \leq 1 \end{cases} \tag{2.27}$$

Consider the right part of the domain ($\frac{1}{2} < x \leq 1$) and insert $u_0'(x_0)$ and $f''(u_0(x_0))$ into the discontinuity condition

$$1 + u_0'(x) f''\left(u_0(x_0)\right) t = 0 \tag{2.28}$$

$$1 + (-2)2t = 0 \tag{2.29}$$

$$1 - 4t = 0 \tag{2.30}$$

$$\Leftrightarrow t = \frac{1}{4} \tag{2.31}$$

Conclude from this that for time $t = \frac{1}{4}$ a first shock forms.

2.3 Rarefaction waves

2.4 Rankine-Hugoniot Condition

Consider the PDE $u_t + f(u)_x = 0$ with the known solution $u(x,t)$. The Rankine-Hugoniot condition yields an, explicit, expression for the speed of a shock (s), given a shock occurs:

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{2.32}$$

with u_l the (constant) value on the left of the shock and u_r the value on the right.

Proof: The Rankine-Hugoniot Conditon $s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$
Consider the following assumptions:
The solution to the scalar conservation problem $u_t + f(u)_x$ is yields a shock
The shock, moving with a positive velocity, along a path is represented by $u_l(t)$ and $u_r(t)$
u_l and u_r are locally constant
The shock speed is represented by $s = \frac{\Delta x}{\Delta t}$
Consider the window $R = [x_1, x_1 + \Delta x] \times [t_1, t_1 + \Delta t]$ around the shock.
Start by integrating the PDE around the shock (the window R) over space and time

$$0 = \iint_R \left(u_t + f(u)_x\right) dx dt \tag{2.33}$$

$$= \int_{t_1}^{t_1+\Delta t} \frac{d}{dt} \int_{x_1}^{x_1+\Delta x} u dx dt + \int_{t_1}^{t_1+\Delta t} f(u)\Big|_{x_1}^{x_1+\Delta x} dt \tag{2.34}$$

$$= \int_{x_1}^{x_1+\Delta x} u\Big|_{t_1}^{t_1+\Delta t} dx + \int_{t_1}^{t_1+\Delta t} f(u)\Big|_{x_1}^{x_1+\Delta x} dt \tag{2.35}$$

Now, consider the assumption that the scalars u_l and u_r are constant in the window R and that R is small. With this evaluate

$$\begin{matrix} u(t_1) = u_r & u(t_1 + \Delta t) = u_l \\ f(u(x_1)) = f(u_r) & f(u(x_1 + \Delta x)) = u_r \end{matrix} \tag{2.36}$$

and simplify the integrals to

$$0 = \int_{x_1}^{x_1+\Delta x} \underbrace{(u_l - u_r)}_{\text{constant}} dx + \int_{t_1}^{t_1+\Delta t} \left(f(u_r) - f(u_l)\right) dt \tag{2.37}$$

$$= (u_l - u_r)\Delta x + \left(f(u_r) - f(u_l)\right) \Delta t \tag{2.38}$$

$$\Leftrightarrow \frac{\Delta x}{\Delta t} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = s \tag{2.39}$$

□

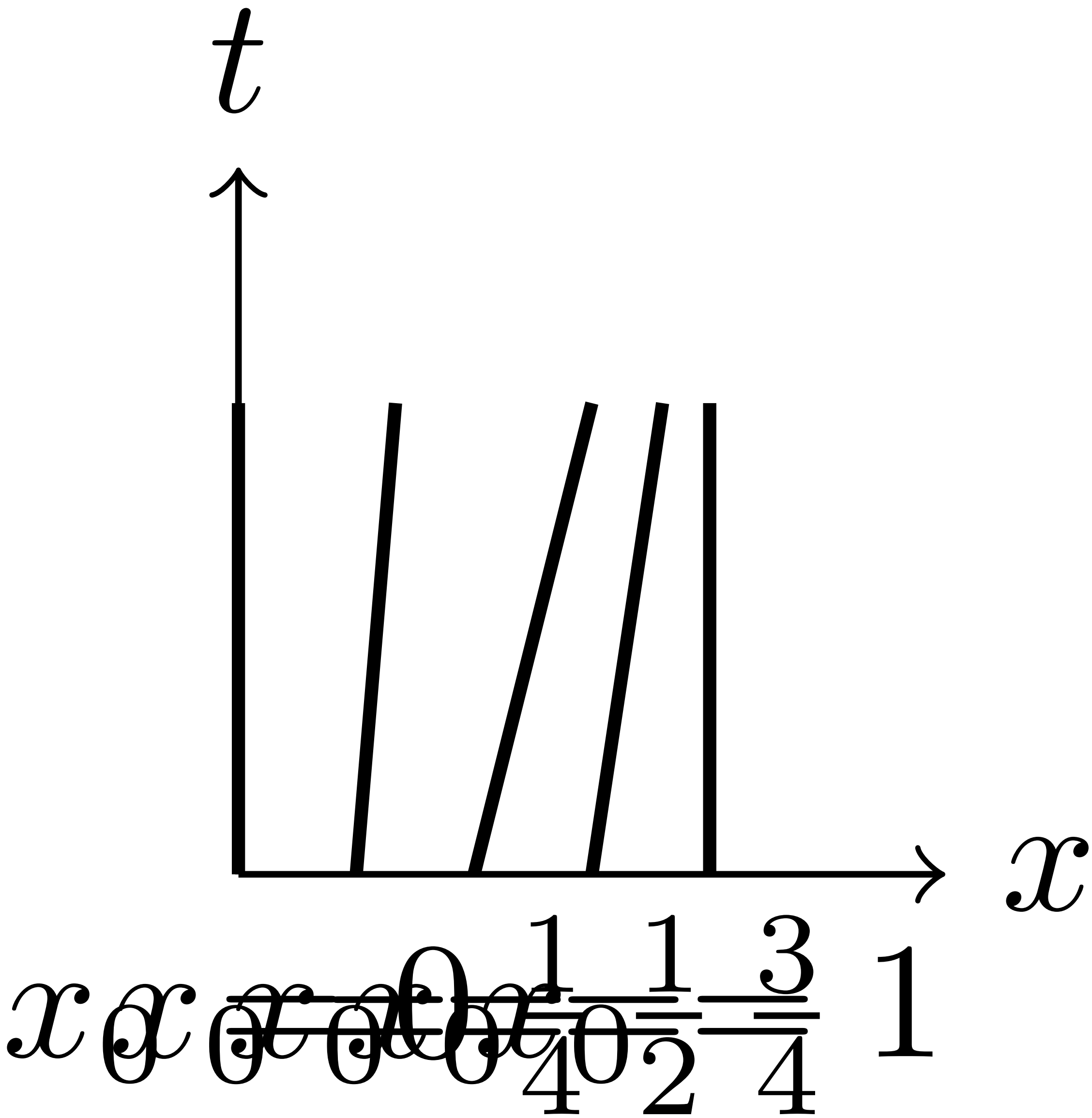

Figure 2.1. Rarefaction and Compression wave

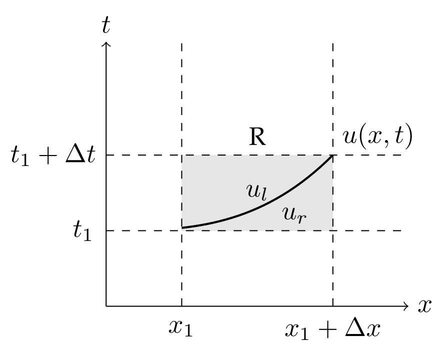

Figure 2.2. Schematic Figure of Rankine-Hugoniot condition window

2.5 The Class of Riemann Problems

A Riemann problem is a specific problem for the scalar conservation law, inspected above, which reads

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,t=0) = u_0(x) \quad = \begin{cases} u_l & , x < 0 \\ u_r & , x > 0 \end{cases} \end{cases} \tag{2.40}$$

For the Riemann problem there can be found two different classes of solutions

1) **Similarity Solution**: A similarity solution connects the two initial states by a continuous solution
2) **Shock Solution:** A shock solution connects u_l and u_r by a discontinuous solution, where the discontinuity moves with the speed s, derived by the Rankine-Hugoniot condition

In the following assume the flux function to be convex. All derivates can be performed for a concave flux function in the same manner, but will produce different results.

2.5.1 Similarity Solution

A continuous similarity solution occurs under the condition $f'(u_l) < f'(u_r)$. For the Riemann problem the solution is undefined for $x = 0$, where two different characteristics, which will not cross.

$$x = f'(u_l)t \qquad x = f'(u_r)t$$

In between these two characteristics, there is one more valid characteristic, namely $x = kt$ with $f'(u_l) < k < f'(u_r)$ for $k = \frac{x}{t}$. The scalar solution is of the form

$$u(x,t) = \begin{cases} u_l & , \frac{x}{t} \leq f'(u_l) \\ v(\frac{x}{t}) & , f'(u_l < x < f'(u_r) \\ u_r & , \frac{x}{t} \geq f'(u_r) \end{cases} \tag{2.41}$$

The arising question is how to calculate $v(\frac{x}{t}) = u(x,t)$. For this, insert the partial derivatives

$$u_t(x,t) = \frac{\partial}{\partial t} v\left(\frac{x}{t}\right) = v'\left(\frac{x}{t}\right)\frac{\partial}{\partial t}\left(\frac{x}{t}\right) = \frac{-x}{t^2}v'\left(\frac{x}{t}\right) \tag{2.42}$$

$$u_x(x,t) = \frac{\partial}{\partial x} v\left(\frac{x}{t}\right) = v'(\left(\frac{x}{t}\right)\frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}v'\left(\frac{x}{t}\right) \tag{2.43}$$

$$f(u)_x = f'(u)u_x = f'\left(v\left(\frac{x}{t}\right)\right)v'\left(\frac{x}{t}\right)\frac{1}{t} \tag{2.44}$$

into the partial differential equation

$$0 = u_t + f(u)_x \tag{2.45}$$

$$= \frac{-x}{t^2}v'\left(\frac{x}{t}\right) + f'\left(v\left(\frac{x}{t}\right)\right)v'\left(\frac{x}{t}\right)\frac{1}{t} \qquad \Big| \cdot t \tag{2.46}$$

$$= v'\left(\frac{x}{t}\right)\left[\frac{-x}{t} + f'\left(v\left(\frac{x}{t}\right)\right)\right] \tag{2.47}$$

This equation can be fulfilled in two ways, either let $v'\left(\frac{x}{t}\right) = 0$. However, this would imply $v\left(\frac{x}{t}\right) = u(x,t)$ to be constant and can only be valid if $u_l = u_r$. The second possibility to fulfill the equation is to ask for

$$\frac{-x}{t} + f'\left(v\left(\frac{x}{t}\right)\right) = 0 \tag{2.48}$$

$$\Leftrightarrow f'\left(v\left(\frac{x}{t}\right)\right) = \frac{x}{t} \tag{2.49}$$

$$\Leftrightarrow (f')^{-1}\left(\frac{x}{t}\right) = (f')^{-1}\left(f\left(v\left(\frac{x}{t}\right)\right)\right) = v\left(\frac{x}{t}\right) \tag{2.50}$$

Where $(f')^{-1}$ is the inverse of the derivative of the flux function. Note that this inverse exists, as the flux function is convex and therefore the second derivative is positive. Now, insert the solution 2.50 into the general solution 2.41 to get the solution

$$u(x,t) = \begin{cases} u_l & , \frac{x}{t} \leq f'(u_l) \\ (f')^{-1}\left(\frac{x}{t}\right) & , f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & , \frac{x}{t} \geq f'(u_r) \end{cases} \tag{2.51}$$

2.5.2 Shock Solution

Contrary to the similarity solution, the shock solution yields a discontinuous solution to the Riemann problem under the condition $f'(u_l) > f'(u_r)$. For the shock solution two characteristics cross, yielding a discontinuous solution with a jump. This jump moves with the shock speed s, derived by the Rankine-Hugoniot condition (equation 2.39). The final solution reads

$$u(x,t) = \begin{cases} u_l & , \frac{x}{t} \leq s \\ u_r & , x < \frac{x}{t} \end{cases} \qquad \text{with: } s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{2.52}$$

2.5.3 Weak Solutions

Weak solutions can simplify mathematical problems by introducing some test function Φ. The original problem will be multiplied by the test function and integrated over the domain Ω. Weak solutions have two important properties

1) **No Derivatives:** Weak solutions involve no derivatives in u and $f(u)$ as they are moved to the test function
2) **Solution Space:** The solution space of weak solutions is much larger then the solution space for strong solutions. Weak solutions can even include more then one solution for a given problem

For a weak solution consider a smooth test function $\Phi(x,t)$ with compact support (the test function is zero outside some finite box). To derive the weak form of the problem multiply the PDE with the test function and integrate over the domain $\Omega = \underbrace{[x_1, x_2]}_{\mathcal{R}} \times \underbrace{[t_1, t_2]}_{\mathcal{R}^+}$.

The weak form then reads: Find u s.t. $\forall \Phi \in C_0^1(\mathcal{R} \times \mathcal{R}^+)$:

$$0 = \iint_\Omega \left[u_t + f(u)_x\right] \Phi(x,t) d\Omega \tag{2.53}$$

$$= \int_0^\infty \int_{-\infty}^\infty \left(u_t \Phi(x,t) + f(u)_x \Phi(x,t)\right) dxdt \tag{2.54}$$

$$= \int_0^\infty \int_{-\infty}^\infty \left(u \Phi_t(x,t) + f(u) \Phi_x(x,t)\right) dxdt + \int_{-\infty}^\infty u_0(x) \Phi(x, t=0) dx \tag{2.55}$$

Note that in the last step integration by parts and the property that the test function vanishes on the boundaries was utilized. Any function $u(x,t)$ that fulfills the weak form (equation 2.55) is a weak solution to the initial given problem. This yields another problem, as the solution u is not necessarily unique. To determine if a weak solution is the correct solution to a specific problem, the (Lax) entropy condition is introduced.

Definition 1. Lax Entropy Condition

A weak shock solution is a strong solution for a problem, if, and only if, the shock satisfies

$$f'(u_l) > s > f'(u_r) \tag{2.56}$$

with the shock speed s. Conclude from this

1) **Jump Solution:** A jump solution is a valid solution if $u_l > u_r$
   a) The flux function is convex and yields $f'(u_l) > f'(u_r)$. Conclude $f'(u_l) > s > f'(u_r)$
2) **Similarity Solution:** The similarity solution is a valid solution if $u_l < u_r$
   a) The flux function is convex and yields $f'(u_l) < f'(u_r)$. Conclude that no discontinuity of the form $f'(u_l) > s > f'(u_r)$ can occur.

3 Implementation

4 Results

5 Discussion and Conclusion

References

Appendix